Journal of Mathematics and Statistics

Original Research Paper

Establishment of Readily Mathematical Formulation for the Evaluation of Slope Stability in Earth-Fill Dams

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Article history
Received: 11-07-2023
Revised: 16-09-2023
Accepted: 19-09-2023

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Abstract: The assessment of slope stability has been considered to be one of the most common issues to deal with by geotechnical engineers. This is due to the amount of destruction that can be brought up by the slope failure of soil masses to roads, railways, and earth dams, for example. Therefore, the stability of soil mass slopes must be carefully analysed prior to, during, and after the construction of such a structure. Depending on the importance of the slope stability problem, a number of alternative methods have been proposed for the evaluation of slope safety. These can generally be divided into two main types: Limit equilibrium methods and finite element methods. In limit equilibrium methods, the soil mass is subdivided into a set of slices with a straight base where each slice must fulfil the equilibrium conditions of forces and/or moments. For the sake of accuracy, a large number of slices are needed to accurately represent the curved slip surface, which is computationally time-consuming. Moreover, the current approaches lack continuity as the inter-slice forces have different slope angles and so they are not equal. The purpose of the study is to develop a mathematical approach that precisely represents the curved slip surface and eliminates the process of slicing. A fast and accurate mathematical formulation that computes the resultant slope forces by just using the geometrics of slip surface and without the need for slicing has been presented in this study. The proposed approach uses the principle of integration with infinitesimal strips that address the issue of inter-slice continuity. Compared with different sets of slices, the new formulation outperformed the ordinary method of slices in terms of accuracy and efficiency.

Keywords: Infinitesimal Slices, Mathematical Formulation, Slope Stability, Limit Equilibrium, Earth-Fill Dam

Introduction

The slope stability of soil materials represents an essential design aspect a geotechnical engineer has to deal with in several civil engineering structures such as earth-fill dams, and road and railway embankments. As most soils exhibit two types of shear resistance forces, namely cohesion and friction, the slip failure surface tends to be rotational and the shape approximates to the arc of a circle. Since there is a large number of failure surfaces for a given slope, a trial search procedure is required to detect the critical slip surface in which the slope safety is at the minimum value. The limit equilibrium slicing method is the most widely used approach by researchers of slope stability because of its well-established and conventional nature (Fellenius, 1936; Bishop, 1955; Janbu, 1954; Morgenstern and Price, 1965; Spencer, 1967; Fredlund and Krahn, 1977; Duncan, 1996). This approach divides the soil into a set of slices and then each slice must fulfil equilibrium conditions of either forces or moments or both of them. The factor of safety is calculated by comparing shear strength along the sliding surface and the required force that can keep the slope in equilibrium.

Clearly, the accuracy of the slicing method is highly dependent on the number of slices making up the slip surface. This is because the base of each slice is assumed to be a straight line, which is not the case in the curved slip surface. The aforementioned assumption has the effect of underestimating the weights of slices, which in turn will underestimate the force components resolved from such weights. In order to eliminate the error in slice weights, a large number of slices is required but this will be at the cost of computational complexity and effort.

In this study, a fast and accurate mathematical formulation that calculates all forces incorporated into the slope stability analysis without the need for slicing the slip surface is presented. Instead of analyzing each slice individually, the new approach simply uses the principle of infinitesimal strips to calculate the area under curves and the tangential slopes of curves. The exact representation of both the area and the slope of the slip surface means that the weight of the slip surface and the directions of weight components are accurately determined. The main goal of this research is the derived general formulae that are based on addressing the issue of continuity between inter-slice forces and exactly calculating the weight, normal force, and tangential force of the whole slip surface in one step. Once the trial surface is available, the derived formulae require just the geometrics of the surface in order to evaluate all surface forces. To demonstrate the performance of the proposed formulation, it was applied to a slip surface problem and compared to the ordinary slicing method using different sets of slices. While the new approach readily and exactly calculates the resultant tangential and normal forces along the slip surface, the slicing method required hundreds of slices to converge to such forces. This has been carried out by gradually increasing the number of slices and comparing the results with the new approach.

The significance of the proposed approach stems from its potential applicability in geotechnical software packages that use finite element methods, which still incorporate stepping analysis procedures. These tools normally start the analysis with a trial slip surface, which can be picked up by the proposed approach, extracting the dimensions and instantly and accurately calculating the forces along the slip surface.

Development of the Method of Slices

The assessment of slope stability is still a challenging and crucial element of geotechnical engineering. There are different computing ways developed to evaluate the safety of slope stability. These, for example, include limit equilibrium methods, finite element methods, finite difference methods, and discrete element methods. Amongst the aforementioned methods, the limit equilibrium method is considered the most common and practical method used in analyzing and predicting the stability of slopes. They are simply based on calculating a single factor of safety determined by the equilibrium between shear stress and shear strength. The slope is considered stable when the factor of safety is greater than unity, which suggests that the forces resisting the slope failure are greater than those driving the failure. When the slope undergoes complex failure mechanisms such as internal deformations or liquefaction processes, more advanced numerical and finite element-based models should be used.

Since all slope stability analysis methods share the concept of the critical slip surface, which is the surface with the minimum factor of safety, the search of the critical surface starts with a trial surface and then an optimization technique is incorporated until a convergence towards the minimum factor of safety is achieved. Once the trial surface is available, the shear strength along the sliding surface is determined using the Mohr-Coulomb expression in all limit equilibrium methods. The shear strength of the soil is defined as the shear stress at which a soil fails in shear. In short-term conditions, the zero-friction approach is used in determining the shear strength while, in long-term conditions, the non-zero friction approach is used. Since the slip surface might pass through different materials such as in zoned earth dams, which means that the angle of shearing resistance is no longer constant, subdividing the slip surface into vertical slices is more appropriate in this situation.

The first method of slices (Fellenius, 1936) referred to as the ordinary method of slices or the Swedish method was introduced for the analysis of circular slip surfaces. This method is based on a linear relationship for the Factor of Safety (FoS). A new relationship for the base normal force with a non-linear equation for FOS was introduced to improve the first method (Bishop, 1955). For non-circular failure surfaces, a simplified method was developed by dividing a potential sliding mass into several vertical slices (Janbu, 1954). Further development of the simplified method while developing the generalized procedure of slices was made (Janbu, 1973). The introduction of the inter-slice forces using different assumptions added further contributions to the previous developments (Morgenstern and Price, 1965; Spencer, 1967; Sarma, 1973). A general procedure of limit equilibrium (Chugh, 1986; Krahn, 2003; Abramson *et al*., 2001) was developed as an extension of the different assumptions (Morgenstern and Price, 1965; Spencer, 1967; Sarma, 1973) based on satisfying both moment and force equilibrium conditions. However, all the above-mentioned developments still require calculating the contributions to resisting and disturbing forces by each slice separately in order to calculate FOS. To date, no general procedure developed for limit equilibrium methods capable of calculating FOS without the need for slicing is available in the literature.

Review of Some of Limit Equilibrium Slice Methods

The Ordinary Method (OM) of slices (Fellenius, 1936) is based on satisfying the moment equilibrium for a circular slip surface but it neglects both the inter-slice normal and shear forces. The advantage of this method is that the equation of the FOS is solved directly and does not require an iteration process as follows:

$$FOS = \frac{\sum_{i=1}^{m} c\,\Delta L_i + \tan\phi \sum_{i=1}^{m} w_i \cos\alpha_i}{\sum_{i=1}^{m} w_i \sin\alpha_i} \quad (1)$$

where, c = soil cohesion in KN/m^2, ΔL_i = base length of slice i in m, ϕ = angle of shearing resistance in degrees, w_i = weight of slice i in KN/m, α_i = angle of inclination of slice base to the horizontal in degrees, m = number of slices making up the slip surface.

Equation (1) produces fewer values of FOS than those yielded by more accurate and advanced methods.

Bishop's Simplified Method (BSM) differs from OM in that it resolves forces in the vertical direction instead of a direction normal to the slip surface. In this way, the inter-slice normal forces are considered in the calculations but still neglecting the inter-slice shear forces. BSM is more common in practice for circular shear surfaces because it produces higher values of FOS than those obtained from OM and is very close to those obtained from more refined methods. The simplified analysis of this method results in:

$$FOS = \frac{\sum_{i=1}^{m}[(c\,\Delta L_i + \tan\phi\, w_i \cos\alpha_i)\, M_a^{-1}]}{\sum_{i=1}^{m} w_i \sin\alpha_i} \quad (2)$$

where,

$$M_a = \cos\alpha_i + \frac{\tan\phi \sin\alpha_i}{FOS} \quad (3)$$

Clearly, solving (Eq. 2) for the value of FOS requires iteration procedures because FOS appears on both sides of the equation. Therefore, computer programs are utilized to solve this equation using numerical methods.

Janbu's Simplified Method (JSM) (Janbu, 1954) was developed for composite slip surfaces based on resolving forces in the horizontal direction. The method is similar to BSM in that it considers inter-slice normal forces but neglects the shear forces. The base normal force is determined in the same way as in BSM and the FOS is computed as follows:

$$FOS = \frac{\sum_{i=1}^{m}[(c\,\Delta L_i + \tan\phi\, w_i \cos\alpha_i)\sec\alpha_i]}{\sum_{i=1}^{m} w_i \tan\alpha_i + \Delta E_i} \quad (4)$$

where, $\Delta E_i = E_2 - E_1$ net inter-slice normal forces for slice i.

To account for inter-slice shear forces, Janbu corrected JSM by introducing a correction factor that gives a lower range for cohesionless (friction only) soils and a higher range for cohesive or clayey soils.

Janbu's Generalized Method (JGM) (Janbu, 1973) is the first method to consider the satisfaction of both force and moment equilibrium. The method takes into account the normal and shearing inter-slice forces by assuming a line of thrust in order to determine a relationship for inter-slice forces. The resulting equation for FOS is a complex function computed by:

$$FOS = \frac{\sum_{i=1}^{m}[(c\,\Delta L_i + \tan\phi\, w_i \cos\alpha_i)\sec\alpha_i]}{\sum_{i=1}^{m}(w_i - \Delta T_i)\tan\alpha_i + \Delta E_i} \quad (5)$$

where, $\Delta E = T_i = T_2 - T_1$ net inter-slice shearing forces for slice i.

Lowe-Karafiath's method (Lowe, 1960) computed the inter-slice resultant force by assuming its inclination as the average of the slope surface inclination (β) and the slice base inclination (α). Despite that the L-KM method considers both inter-slice normal and shear forces, it does satisfy the force equilibrium only. Similar to the L-KM method, the US Army Corps of Engineers (1982) method assumes the inter-slice force inclination in two ways: It can be assumed either parallel to the ground surface or equal to the average slope angle between the entry and exit points of the critical slip surface. Sarma (1973) was the first to develop a method for a non-vertical slice or general blocks. This method satisfies both moment and force equilibrium in addition to relating the interslice forces by a quasi-shear equation. The Morgenstern-Price method (Morgenstern and Price, 1965) or M-PM suggests assuming a function for the inter-slice force of any type like half-sine, trapezoidal, or user-defined. For a given force function, an iteration procedure is required to compute the inter-slice forces. Spencer's method (Spencer, 1967) is similar to M-PM but it differs in the assumption of constant inclination of inter-slice forces.

Mathematical Formulation of the Proposed Approach

The main shortcoming of slice methods is the assumption that, for all slices, the resultant of inter-slice forces is inclined at an angle parallel to the base of the slice. This assumption does not satisfy inter-slice equilibrium because all adjacent slices have different base inclination angles. In addition, the assumption of a straight base underestimates the weight of each slice and consequently underestimates the tangential and normal forces. The only condition in which adjacent slices have similar inclination angles takes place when each slice has an infinitesimal width (dx), for which no method has been yet developed to date. This is the key to the development of the proposed approach, which eliminates the assumption of inter-slice resultant force inclination. Figure 1 shows a mathematical representation of a sloped soil embankment with a side slope of H: B, in which H represents dam height and B represents the slope base length. A circular slip surface (AE) is defined by radius (r) and center (O) located at coordinates (a, b) with respect to the origin at point (A). The slip surface is subdivided into an infinite number of infinitesimal slices each with width (dx), base length (ΔL), and base inclination angle (α) with the horizontal axis (X).

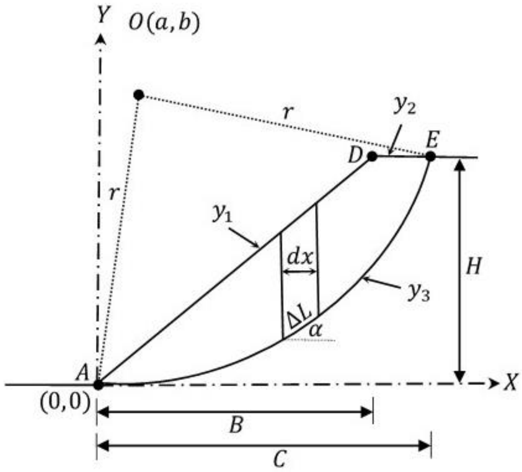


Fig. 1: Mathematical representation of the variables defining the slip surface

Mathematically, the equations describing the boundaries of the slip surface can be written in terms of geometric dimensions of the embankment and the slip surface as follows:

$$y_1 = \frac{H}{B}x \tag{6}$$

$$y_2 = H \tag{7}$$

$$y_3 = b - \sqrt{r^2 - (x-a)^2} \tag{8}$$

If the embankment is made up of homogeneous soil with unit weight (γ_s), then the weight of the infinitesimal slice denoted with (dw) can be computed for part (A, D) by:

$$dw = \gamma_s \left(\frac{H}{B}x - b + \sqrt{r^2 - (x-a)^2}\right) dx \tag{9}$$

and for the part (D, E) by:

$$dw = \gamma_s \left(H - b + \sqrt{r^2 - (x-a)^2}\right) dx \tag{10}$$

The inclination of the infinitesimal slice base can be mathematically found from the definition of tangential slope (dy_3/dx) as follows:

$$\tan\alpha = \frac{(x-a)}{\sqrt{r^2 - (x-a)^2}} \tag{11}$$

From which the trigonometric expressions sin α and *cos* α can be written as:

$$\sin\alpha = \frac{(x-a)}{r} \tag{12}$$

$$\cos\alpha = \frac{\sqrt{r^2 - (x-a)^2}}{r} \tag{13}$$

The length of the infinitesimal slice base ($\Delta L = dx/\cos\alpha$) can be expressed by:

$$\Delta L = \frac{r}{\sqrt{r^2 - (x-a)^2}}\, dx \tag{14}$$

Now, the normal ($dN = dw \cos\alpha$) and shear ($dT = dw \sin\alpha$) forces at the base of the infinitesimal slice can be computed for part (A, D) by:

$$dN = \gamma_s \left(\frac{H}{B}x - b + \sqrt{r^2 - (x-a)^2}\right) \frac{\sqrt{r^2 - (x-a)^2}}{r}\, dx \tag{15}$$

$$dT = \gamma_s \left(\frac{H}{B}x - b + \sqrt{r^2 - (x-a)^2}\right) \frac{(x-a)}{r}\, dx \tag{16}$$

while for part (D, E) by:

$$dN = \gamma_s \left(H - b + \sqrt{r^2 - (x-a)^2}\right) \frac{\sqrt{r^2 - (x-a)^2}}{r}\, dx \tag{17}$$

$$dT = \gamma_s \left(H - b + \sqrt{r^2 - (x-a)^2}\right) \frac{(x-a)}{r}\, dx \tag{18}$$

The total weight of the slip surface ($W = \sum dw$), total normal force ($N = \sum dN$), total shear force ($T = \sum dT$), and length of the slip surface ($L_a = \sum \Delta L$) can be computed by:

$$W = \gamma_s \int_0^B \left(\frac{H}{B}x - b + \sqrt{r^2 - (x-a)^2}\right) dx + \gamma_s \int_B^C \left(H - b + \sqrt{r^2 - (x-a)^2}\right) dx \tag{19}$$

$$N = \gamma_s \int_0^B \left(\frac{H}{B}x - b + \sqrt{r^2 - (x-a)^2}\right) \frac{\sqrt{r^2 - (x-a)^2}}{r}\, dx + \gamma_s \int_B^C \left(H - b + \sqrt{r^2 - (x-a)^2}\right) \frac{\sqrt{r^2 - (x-a)^2}}{r}\, dx \tag{20}$$

$$N = \gamma_s \int_0^B \left(\frac{H}{B}x - b + \sqrt{r^2 - (x-a)^2}\right) \frac{\sqrt{r^2 - (x-a)^2}}{r}\, dx + \gamma_s \int_B^C \left(H - b + \sqrt{r^2 - (x-a)^2}\right) \frac{\sqrt{r^2 - (x-a)^2}}{r}\, dx \tag{21}$$

$$L_a = \int_0^C \frac{r}{\sqrt{r^2 - (x-a)^2}}\, dx \tag{22}$$

DOI: 10.3844/jmssp.2023.37.43

Equations (19-20) can be easily determined either numerically or mathematically using the basic rules of integration. For example, the resulting expressions for (W) and (L_a) are:

$$W = \gamma_s\left[\frac{HB}{2} - bB + H(C-B) - b(C-B) + \frac{a}{2}\sqrt{r^2-a^2} + \frac{\pi r^2}{360}\sin^{-1}\frac{a}{r} + \frac{C-a}{2}\sqrt{r^2-(C-a)^2} + \frac{\pi r^2}{360}\sin^{-1}\frac{C-a}{r}\right] \quad (23)$$

$$L_a = \frac{\pi r}{180}\left[\sin^{-1}\frac{C-a}{r} + \sin^{-1}\frac{a}{r}\right] \quad (24)$$

Similarly, the resulting expressions for (T) and (N) are:

$$T = \gamma_s\left[\frac{H}{Br}\left(\frac{B^3}{3} - \frac{aB^2}{2}\right) + \frac{1}{r}\left(\frac{(r^2-a^2)^{\frac{3}{2}}}{3}\right) + \frac{H}{r}\left(\left(\frac{C^2}{2} - aC\right) - \left(\frac{B^2}{2} - aB\right)\right) - \frac{b}{r}\left(\frac{C^2}{2} - aC\right) - \frac{1}{r}\left(\frac{(r^2-(C-a)^2)^{3/2}}{3}\right)\right] \quad (25)$$

$$\begin{aligned} N = \frac{H}{Br}\Bigg[&\frac{a(B-a)}{2}\sqrt{r^2-(B-a)^2} \\ &+ \frac{a\pi r^2}{360}\sin^{-1}\frac{B-a}{r} - \frac{(r^2-(B-a)^2)^{3/2}}{3} + \frac{a^2}{2}\sqrt{r^2-a^2} + \frac{a\pi r^2}{360}\sin^{-1}\frac{a}{r} + \frac{(r^2-a^2)^{\frac{3}{2}}}{3}\Bigg] \\ &- \frac{b}{r}\left[\frac{a}{2}\sqrt{r^2-a^2} + \frac{\pi r^2}{360}\sin^{-1}\frac{a}{r}\right] - \frac{a^3}{3r} \\ &+ \frac{H}{r}\left[\frac{(C-a)}{2}\sqrt{r^2-(C-a)^2} + \frac{\pi r^2}{360}\sin^{-1}\frac{C-a}{r} - \frac{(B-a)}{2}\sqrt{r^2-(B-a)^2} - \frac{\pi r^2}{360}\sin^{-1}\frac{B-a}{r}\right] - \frac{b}{r}\left[\frac{(C-a)}{2}\sqrt{r^2-(C-a)^2} + \frac{\pi r^2}{360}\sin^{-1}\frac{C-a}{r}\right] + rC \\ &- \frac{(C-a)^3}{3r} \end{aligned} \quad (26)$$

Clearly, the above expressions (Eqs. 23-26) are dependent only on the geometrics of the embankment slope and the slip surface and do not require any calculations of slice forces, which are dependent on the dimensions and inclination angle for each slice. The proposed method directly and accurately calculates the total slice forces once the slip surface is available without the need to slice the slip surface. This gives the advantage of calculating FOS directly from the geometrics of the slip surface. For example, no requirement for measuring the base inclination calculating the slice forces, or even calculating the weights of slices.

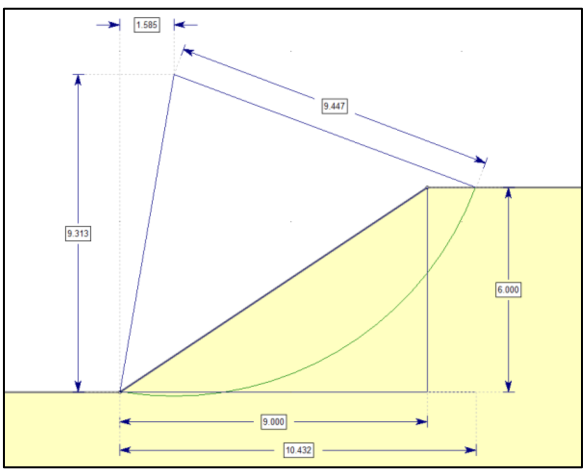


Fig. 2: Application example (All dimensions are meters)

Application Example

To demonstrate the performance of the proposed method, a soil embankment representing part of an earth dam is presented in Fig. 2. The embankment is made up of homogeneous soil with a unit weight of 20 KN/m^3 and cohesion of 10 KN/m^2. The angle of shearing resistance of the soil is 29°. The embankment is in drained condition so it neglects the effect of pore water pressure. The embankment slope has a height of 6 m and a base length of 9 m. The slip surface has a radius of 9.447 m starting from the toe and extends horizontally for a distance of 10.432 m to intersect with the embankment crest.

Results and Discussion

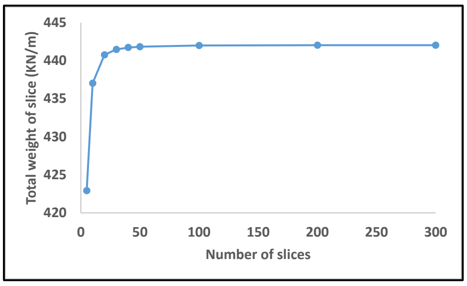
The proposed method is applied using the geometrics of the slope, which gives the following slope properties: $a = 1.585\ m$, $b = 9.313\ m$, $r = 9.447\ m$, $B = 9\ m$, $C = 10.432\ m$, and $H = 6\ m$. These values are sufficient to calculate the total slip surface forces and evaluate the stability of the surface using the developed method. The resulting total normal and shearing slice forces were found directly from (Eqs. 25-26) at 197.808 and 373.864 KN/m respectively, while the total weight and the length of the slip surface were calculated directly from (Eqs. 23-24) at 442.039 KN/m and 13.046 m respectively. Accordingly, the value of FOS using (Eq. 1) was 1.707.

In contrast, the results of applying OM using different sets of slices are shown in Table 1. A geotechnical software, namely Slide, was utilized to analyze the various cases of slicing using OM. Evidently, the method of slices underestimates the slice forces and it requires a large set of slices in order to converge to the exact values of forces. The underestimation of slip surface weight (W) means an underestimation of both the sliding force (T) and the resisting force (N). More importantly, the method of slices produces larger values of FOS if small sets of slices are used. This indicates that OM overestimates the safety indicator (FOS) in real problems. Besides, the method of slices needs to analyze each slice individually before summing up the total slice resultant forces. In other words, the computational effort is high because it incorporates a series of calculation procedures like sizing, calculation of base inclination, and force resolving for each slice. The developed method eliminates such procedures by applying a block analysis for the whole slip surface just using the geometric dimensions, which are readily available once the trial surface is provided.

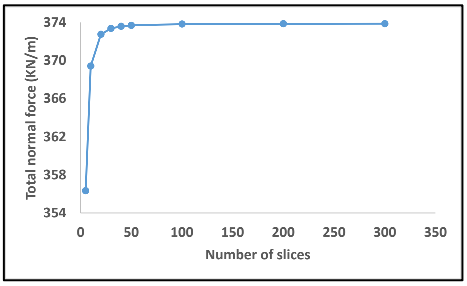
The novelty of the new formulation stems from the instant calculation of total slip surface forces by just reading the dimensions of the given surface. No incorporation of slices in any way and the formulation is an exact representation of the slip surface in terms of force analysis. The determination of slip surface forces in one go is actually a significant step in the context of slope stability analysis in earth-fill dams.

Table 1: Results of slice forces using different sets of slices

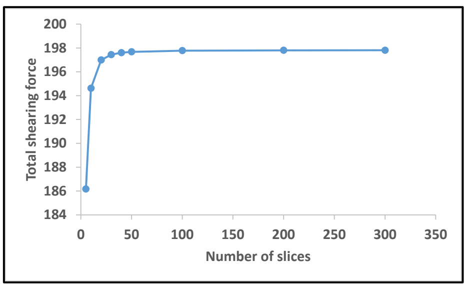
Number of slices	W (KN/m)	T (KN/m)	N (KN/m)	FOS
5	422.913	356.329	186.170	1.759
10	437.033	369.416	194.621	1.722
20	440.768	372.744	196.992	1.711
30	441.475	373.370	197.444	1.709
40	441.724	373.588	197.604	1.708
50	441.839	373.689	197.679	1.708
100	441.994	373.824	197.779	1.707
200	442.032	373.858	197.803	1.707
300	442.039	373.864	197.808	1.707



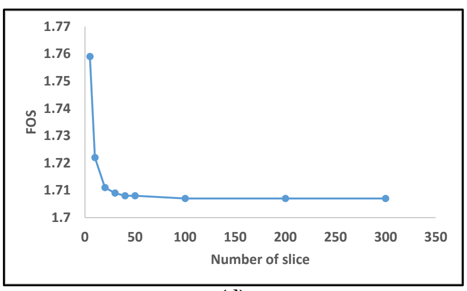
(a)



(b)



(c)



(d)

Fig. 3: (a) Relationship between number of slices and total weight; (b) Total normal force; (c) Total shearing force; (d) FOS

Figures (3a-d) show how the number of slices is related to the various slice properties. All of the relationships showed an agreement that more than 50 slices are required for the OM to start converging to the exact values, which is calculated in a one-step procedure using the developed method. Moreover, the slip surface was subdivided into a number as large as 300 slices in order to produce similar results to those obtained with the proposed method.

Conclusion

A fast and accurate mathematical formulation that produces readily mathematical equations for the calculation of resultant forces of soil slopes was presented. The significant contribution added to the subject of slope stability analysis is represented in eliminating the need to slice the slip surfaces. By using infinitesimal slices, the assumption of equal base inclination for adjacent slides becomes feasible, which was not applicable among the various traditional methods of slices developed previously. From the geometrics of the slope and slip surface, the developed method is able to directly calculate the resultant forces without the need to slice and sum up the slice forces. Since the developed method was formulated for drained homogeneous soils, it is recommended to extend the method to include undrained conditions and zoned soil embankments. This will enhance the study by incorporating both the dry and wet conditions of earth-fill dams. Further study is required to investigate the efficiency of the proposed method against finite element methods.

Acknowledgment

The work presented in this study is a result of cooperation between the University of Tripoli and the University of Zintan, Libya. The role of the first author was in the stability analysis, while the second author was mainly involved in the mathematical formulation. The authors gratefully acknowledge the considerable and administrative support from all contributors.

DOI: 10.3844/jmssp.2023.37.43

Funding Information

This research is the authors' own work and did not receive any form of funding either locally or internationally. We appreciate the support received from the Science Publication to which the publication of our article is attributed.

Author's Contributions

Salah Saleh: Formulation of the concept of the proposed method.

Majda Ibobakr Almhdi Amer: Assistance in the mathematical formulation of the proposed method.

Ethics

The authors ensure observing the standard ethics of research and they are responsible for all of the information presented in this research.

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