Journal of Mathematics and Statistics

Original Research Paper

An Empirical Study of Robust Modified Recursive Fits of Moving Average Models

[1]Mohamed Ali Ismail, [2]Hend A. Auda, [3]Joe W. McKean and [2]Mahmoud Mohamed Sadek

[1]*Department of Statistics, Cairo University, Egypt*
[2]*Department of Statistics, Helwan University, Egypt*
[3]*Department of Statistics, Western Michigan University, USA*

Article history
Received: 11-11-2021
Revised: 28-02-2022
Accepted: 09-03-2022

Corresponding Author:
Mahmoud Mohamed Sadek
Department of Statistics,
Helwan University, Egypt
Email:
mahmoud_mohamed@commerce.helwan.edu.eg

Abstract: The time-series Moving Average (MA) model is a nonlinear model; see, for example. For traditional Least Squares (LS) fits, there are several algorithms to use for computing its fit. Since the model is nonlinear, Fuller discusses a Newton-type step algorithm. proposed a recursive algorithm based on a sequence of three linear LS regressions. In this study, we robustify Koreisha and Pukkila's algorithm, by replacing these LS fits with robust fits. We selected an efficient, high breakdown robust fit that has good properties for skewed as well as symmetrically distributed random errors. Other robust estimates, however, can be chosen. We present the results of a simulation study comparing our robust modification with the Maximum Likelihood Estimates (MLE) in terms of efficiency and forecasting. Our robust modification has relatively high empirical efficiency relative to the MLE estimates under normally distributed errors, while it is much more efficient for heavy-tailed error distributions, including heavy-tailed skewed distributions.

Keywords: High Breakdown, Innovative Substitution, Robust Recursive Algorithm, Robust Modification Fits, Maximum Likelihood, Monte Carlo, Skewed Errors, Moving Average

Introduction

The Moving Average (MA) model is one of the simplest and easily interpretable time series models. For these models, the observed response is modeled as a linear combination of coefficients and lagged random errors (shocks). Although it is easy to describe, this model is nonlinear in its coefficients and is not that easy to fit. In contrast, the Autoregressive (AR) time-series model is a linear combination of coefficients and lagged observations. Thus, the AR model is linear in its coefficients and can easily be fit by any regression procedure. (Fuller, 1996) discusses the usual nonlinear, Least Squares (LS) Gauss-Newton algorithm for fitting a MA model as well as the Maximum Likelihood (MLE) under the assumption of normally distributed random errors. (Hannan and Rissanen, 1982) developed a recursive algorithm for the fit of an MA. (Koreisha and Pukkila, 1989) proposed the Innovative Substitution (IS) recursive algorithm for the fit of an MA. This IS recursive algorithm depends on a series of LS fits and is easy to fit. The authors of these papers showed that the estimates obtained by their recursive algorithms are asymptotically equivalent to the MLEs.

In this study, we consider the empirical properties of simple robust modifications to the IS recursive algorithm (Koreisha and Pukkila, 1989). It involves replacing the LS fits of the IS algorithm with robust fits. We call these estimators Modified Innovative Substitution (MIS) estimators. We have selected a robust fit with several properties in mind. We want the robust fit to have high efficiency for normally distributed random errors as well as for contaminated distributions. The lagged error structure in a MA model implies that outliers in the Y-space (response space) will become outliers in the X-space (fitting-space); therefore, high breakdown robust estimates are preferred. Also, in practice, the distribution of the random errors is often skewed, so the robust estimator should have good efficiency properties under both symmetric and asymmetric random error distributions. One such estimator is the high breakdown rank-Based

(HBR) estimator developed by Chang *et al*. (1999). Its properties for AR models were developed by Terpstra *et al*. (2000). In the empirical study of AR models discussed in (Terpstra *et al*., 2000), the HBR was one of the best estimators in terms of empirical efficiency over a wide range of random error distributions, including symmetric as well as asymmetric random error distributions.

The main goal of this study is the empirical properties of the MIS estimators, so we have conducted two large Monte Carlo studies on the efficiency of estimation and forecasting, comparing the MIS, IS, and MLE procedures. The studies are over MA models of orders one through four, from small ($n = 50$) to large ($n = 500$) sample sizes and for several settings of scale. A variety of error distributions are used, including the normal, contaminated normal, and skewed contaminated normal. Three levels of contamination are used: 10, 20, and 30%. Both Innovative (IO) and Additive (AO) outlier situations are considered. These studies confirm the efficiency of the MIS estimates over this wide range of MA models and random error distributions. In contrast, the traditional estimators, IS and MLE, have poor efficiency over all the non-normal distributions considered. The study also shows that of these two nonrobust estimators, the MLE has better empirical efficiency than the IS estimator over all of the situations.

We used the R package hbrfit for the computations in this study. It is freely downloadable at the site https://github.com/kloke/hbrfit.

Estimates for Moving Average Models

Let Y_t, $t = 1, 2, ..., n$, denote a stationary time-series of length n. In this study, we focus on the Moving Average model of order q, MA(q), which is defined below in expression (4). In this section, we present the Modified Innovative Substitution (MIS) estimators of the coefficients of a Moving Average model (MA). The algorithm that we are modifying for the fit of an MA model depends on an Initial Autoregressive (AR) fit. So we briefly present the AR model first.

Autoregressive Time Series Models

We say that Y_t follows a stationary autoregressive time-series model of order p, denoted here by AR(p), if:

$$Y_t = \phi_0 + \phi_1 Y_t -_1 + \phi_2 Y_t -_2 + \cdots + \phi_p Y_t -_p + e_t \tag{1}$$

where, ϕ_0, ..., ϕ_p are the AR parameters that satisfy the stationarity assumption, i.e., the solutions to the following equation:

$$x^p - \phi_1 x^{p-1} - \phi_2 x^{p-2} - \cdots - \phi_p = 0 \tag{2}$$

Lie in the interval $(-1,1)$; (Box *et al*., 2008).

Computationally Model (1) is a linear model with Y_t as the tth response and the lagged Y's, Y_{t-j}, $j = 1, ..., p$ as the predictors. Hence, robust estimates and subsequent analyses for the $AR(p)$ are easily computed.

There has been considerable work done on robust fits of AR models. For rank-based fits, (Koul and Saleh, 1993) developed the asymptotic theory for these rank-based estimates. Because of the autoregressive model, error distributions with even moderately heavy tails produce outliers in factor space (points of high leverage). With this in mind, the high breakdown rank-based (HBR) estimates of (Chang *et al*., 1999) seem more appropriate. HBR estimates for the coefficients of AR(p) models are developed in (Terpstra *et al*., 2000). These articles present the asymptotic theory for the HBR estimates for AR(p) models; useful practical diagnostics including Studentized residuals and diagnostics which differentiate among the HBR and other fits; and the results of Monte Carlo studies which show that the robustness and efficiency of the rank-based estimators hold for finite sample sizes. More discussion of the HBR is presented in the following section.

Estimators for Moving Average Models

The moving average model of order q is defined as:

$$Y_t = \epsilon_t + \theta_1\epsilon_{t-1} + \theta_2\epsilon_{t-2} + \cdots + \theta_q\epsilon_{t-q}, \tag{3}$$

Where, $\theta_i, i = 1,2,...,q$ are the MA model parameters and ϵ_t are uncorrelated random variables with mean 0 and variance $\sigma^2 < \infty$. This can be equivalently written in terms of the backshift operator B as:

$$Y_t = \theta_q(B)\epsilon_t, \tag{4}$$

where:

$$\theta_q(B) = \left(1 + \theta_1 B + \cdots + \theta_q B^q\right), \tag{5}$$

The t^{th} error of this series can be written as:

$$\epsilon_t = Y_t - \left(\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\right)\epsilon_t. \tag{6}$$

Moving average models are said to be invertible if they can be expressed as an infinite order autoregressive model. This requires that the roots of the characteristic equation:

$$m^q + \theta_1 m^{q-1} + \theta_2 m^{q-2} + \cdots + \theta_q = 0 \tag{7}$$

are less than one in absolute value; see, for example, Chapter 2 of (Fuller, 1996). For examples, the MA (1) model is invertible if $-1 < \theta_1 < 1$ while the MA (2) model is invertible if $-1 < \theta_2 < 1$, $\theta_2 + \theta_1 > -1$, $\theta_1 - \theta_2 < 1$.

DOI: 10.3844/jmssp.2022.87.100

As discussed (Fuller, 1996), the MA model is nonlinear in terms of the parameters θ_i, $i = 1, ..., q$. Fuller discusses fitting the MA model by using a Gauss-Newton type algorithm. He also discusses an algorithm for computing the Maximum Likelihood Estimates (MLE), assuming normally distributed errors. (Hannan and Rissanen, 1982) developed a recursive linear algorithm for estimating the MA coefficients. (Koreisha and Pukkila, 1989) proposed a similar recursive linear algorithm (INNOVATIVE SUBSTITUTION (IS)). Their recursive method is obtained by applying Ordinary Least Squares (OLS) methods to the MA model after replacing lagged errors with the corresponding lagged residuals from an initial long autoregressive fit. The algorithm is given by the following four steps:

1. Obtain the estimated residuals using the autoregressive $\left(AR\left(\sqrt{n}\right)\right)$ model of order $\left(\sqrt{n}\right)$
2. Regress the estimated residuals from the previous step on the observations using the linear regression model
3. Use the estimated regression coefficient from the last step as a θ value to estimate the errors recursively using Eq. (6)
4. The estimates of the errors of Step 3 are plugged into the model as predictor variables and the observations are regressed on these predictors for the final estimates

As shown in our simulation study there is very little loss in empirical efficiency between these IS estimates and the maximum likelihood estimates when the errors have a normal distribution. Similar to the MLE estimates, though, our study shows that the IS estimates are quite sensitive to errors with heavy-tailed distributions. To overcome these problems we propose two modified innovative substitution methods. The first Modified IS estimator, (MIS1), consists of replacing the LS fit in Step 4 with a robust fit, while the second, (MIS2), replaces all the LS regressions in the algorithm with the corresponding robust fit. Our empirical results show that these modified estimators yield estimates which lose little efficiency in comparisons to either the IS or MLE fits when the random errors have a normal distribution and, further, they are much more efficient than either of these estimates are when the random errors have distributions with heavier tails than that of the normal distribution.

For the choice of a robust estimator, consider a moving average model when the error distribution has heavy tails. Due to the MA model's model dependency on the past, outliers generated from heavy-tailed distributions become incorporated in both the responses and predictors. Hence, high breakdown estimates are called for. On the other hand, high efficiency is critical. So, high breakdown estimates which retain efficiency are preferable. Finally, our study is also concerned with the behavior of the estimates under skewed heavy-tailed distributions. So robust high breakdown estimators which are efficient under skewed as well as symmetric distributions are desirable.

One estimator that satisfies all three of these properties is the HBR estimator proposed by Chang *et al*. (1999). This is a weighted Wilcoxon estimator using Schweppe-type weights which are functions of both design (predictor) space and response space. In the design space, the weights are functions of the robust distances based on the minimum covariance determinant (MCD) proposed by (Rousseeuw and Van Driessen, 1999), and in the response space, they are functions of residuals based on an initial Least Trim Squares (LTS) fit. As shown by Chang *et al*. (1999) the HBR estimates achieve a 50% breakdown point. Further, its influence function is everywhere continuous, bounded, and goes to zero in all directions of the spaces of the predictors and responses. So, it is a high breakdown, robust estimator. Since points of "good" leverage are not necessarily down-weighted, the HBR estimates recovery efficiency that is lost by high breakdown fits whose weights depend only on design space; see the results of the empirical study by Chang *et al*. (1999). The HBR estimator performed well in the empirical study concerning robust estimators for autoregressive time-series conducted by Terpstra *et al*. (2000). It was among the best estimators over many situations for heavy-tailed error structure, Including Innovative (IO) and Additive (AO) outliers and skewed as well as symmetric heavy-tailed error distributions.

Our proposed MIS1 procedure is then to use the HBR estimator in the last step of the algorithm instead of the LS estimator. It would seem that replacing all the LS fits in the algorithm with HBR fits would lead to a more efficient estimator in the case of heavy-tailed error distributions. This estimator is our MIS2 estimator.

In the next two sections, we present the results of two Monte Carlo studies comparing the MIS1, MIS2, IS, and MLE estimators. The first study investigates the relative efficiencies of the estimators while the second study compares their forecasts.

Monte Carlo Study of Relative Efficiencies

The purpose of this Monte Carlo study is to determine empirically how well the estimators MLE, IS, MIS1 and MIS2 compare in terms of efficiency over situations determined by the three factors: (I) Moving Average Model, (II) Error Distributions, and (III) Sample Sizes. We discuss the setup of our study and then proceed with the results.

Setup of the Monte Carlo Study

For our Monte Carlo study, the levels of the factors (I) Moving Average Model, (II) Error Distributions, and (III) Sample Sizes are:

(I) The MA models

(a) MA (1): There are 3 MA (1) models with respective coefficients $\theta_1 = 0.2$, $\theta_1 = 0.5$ and $\theta_1 = 0.9$
(b) MA (2): There are 3 MA (2) models with the respective vector of thetas, $(\theta_1,\theta_2) = (0.1,0.2)$, $(\theta_1,\theta_2) = (0.3,0.4)$ and $(\theta_1,\theta_2) = (0.8,0.1)$
(c) MA (3): There are 3 MA (3) models corresponding to the respective vector ofthetas, $(\theta_1,\theta_2,\theta_3) = (0.0,0.0,0.2)$, $(\theta_1,\theta_2,\theta_3) = (0.0,0.0,0.5)$ and $(\theta_1,\theta_2,\theta_3) = (0.0,0.0,0.9)$
(d) MA (4): There are 3 MA (4) models corresponding to the respective vector ofthetas, $(\theta_1,\theta_2,\theta_3,\theta_4) = (0.0,0.0,0.0,0.2)$, $(\theta_1,\theta_2,\theta_3,\theta_4) = (0.0,0.0,0.0,0.5)$ and $(\theta_1,\theta_2,\theta_3,\theta_4) = (0.0,0.0,0.0,0.9)$

(II) The Error Distributions

N Normal with σ selected from: {0.5, 1, 2}. We consider different values of σ, because the MA model is nonlinear and, hence, the estimators may not be equivariant.

CN Symmetric Contaminated Normal. The random error e is of the form:

$$e = (1 - I_\epsilon)Z + I_\epsilon \sigma W, \tag{8}$$

Where, I_ϵ has a Bernoulli distribution with a probability of success ϵ; W and Z have $N(0,1)$ distributions; and I_ϵ, W, and Z are independent. For the parameters, σ is set at 10, while ϵ is selected from {0.10, 0.20, 0.30}.

This is a heavy-tailed error distribution. Based on the lag structure of the MA model, the outliers generated from this distribution become incorporated into the model. These often become "good" points of high leverage. Sometimes these outliers are called Innovative Outliers (IO). This is true of the next distribution, also.

SCN: Skewed Contaminated Normal. The random error e is of the form:

$$e = (1 - I_\epsilon)Z + I_\epsilon \sigma W, \tag{9}$$

Where, I_ϵ has a Bernoulli distribution with a probability of success ϵ; Z has an $N(0, 1)$ distribution; W has an $N(10, 1)$ distribution; and I_ϵ, W, and Z are independent. For the parameters, σ is set at 10, while ϵ is selected from {0.10, 0.20, 0.30}

NAO: This distribution contains Additive Outliers (AO) generated as discussed in (Terpstra *et al.*, 2000). The underlying process is an $N(0, \sigma 2)$ distribution where σ is selected from {.5,1,2}. The outliers are generated at a rate of 0.2 n from an $N(30,1002)$ distribution

SCNAO: This is also a distribution with additive outliers. The underlying distribution is a skewed contaminated normal distribution described in the situation (SCN). In this case, the outliers are generated at a rate of 0.2 n from a skewed contaminated normal distribution with a mean of 100, the standard deviation of 20, and a level of contamination of 30%

(III) Sample sizes are: {50,100, 200, 500}

For each combination of these levels, we have run 1000 simulations for all four estimators: MLE, IS, MIS1 and MIS2.

Results

We compare the MLE, IS, MIS1, and MIS2 estimators using the following metric:

For each estimator and situation, we obtain the mean square error (MSE). As our measure of efficiency, we report the ratio of the MSE of MLE to MSEs of the estimator (MIS1, MIS2, or IS).

For each situation, this ratio is an estimate of the ARE between the estimators, so we label them as AREs in the tables reporting the results.

Comparisons in Terms of Empirical Efficiency

Tables 1-5 present the results of the empirical AREs of the three estimators' overall distributions and situations in the study. For each situation, these are the ratios of the MSEs of the estimators. The ratio is always of the form MSE of the maximum likelihood estimators (MLE) to the MSE of the estimator (MIS1, MIS2, or IS). Hence, values of a ratio less than 1 favor the MLE while values greater than 1 favor the estimator (MIS1, MIS2, or IS). We discuss the results by distributions.

The results for the normal situations are found in the top panel of Tables 1-4 for the MA (1)-MA (4) models, respectively. Naturally, the MLE estimator outperforms the MIS1 and MIS2 estimators for these models. Generally, though, the MIS1 and MIS2 estimators have empirical efficiencies above 90%. For example, over all of the normal situations, the MIS1 estimator has empirical efficiency ≥ 92% for 90 of the possible 108 situations. In several situations, the MIS1 estimator has higher efficiency than the IS estimator. In general, in most situations, the MIS1 estimator has a slightly higher efficiency than the MIS2 estimator. Although the efficiency of the IS generally close to 1, the MLE estimator is more efficient for each situation.

The bottom two panels of Tables 1-4 contain the results for the two innovative outlier distributions. The middle panel contains the results for the symmetric contaminated normal distribution. Over all of these situations, the MIS2 estimator is much more efficient

DOI: 10.3844/jmssp.2022.87.100

than the MLE estimator. Generally, it's between 5 (500%) and 8 (800%) times more efficient than the MLE estimator. For these situations, the MIS2 estimator is always more efficient than the MIS1 estimator, although, generally, the edge is slight. Between the IS and MLE estimators, only in one situation is the IS estimator more efficient. The efficiency of the IS estimator, however, is generally close to 1. The bottom panel of Tables 1-4 displays the empirical efficiencies for the skewed contaminated normal situations. While the trends seen for the symmetric situations are the same for the skewed situations, the MIS2 efficiency is generally larger than in the symmetric case. The MIS2 is generally between 8 to 12 times more efficient than the MLE estimator. The MIS1 estimator is almost as efficient as the MIS2 estimator. Of the two poorest estimators, the MLE is more efficient than the IS estimator.

Table 5 displays the results for the Additive Outlier (AO) distributions. The left side the of table displays the efficiencies for the symmetric AO situations. For these situations, the MIS2 estimator is much more efficient than the MLE estimator. In many situations, it is over 100 times more efficient than the MLE estimator. The MIS2 estimator outperforms the MIS1 estimator in every situation. For the skewed AO situations, right side of Table 5 the results are generally the same.

Table 1: Summary of empirical ARE results of the MA(1) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels for the procedures MIS1, MIS2, and IS. The tabled entries are the ratios of MSE for the MLE estimates divided by the MSE of the procedure.

	θ	σ	0.5			1			2		
Normal		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	0.2	50	0.912	0.870	0.977	0.871	0.857	0.979	0.933	0.905	0.977
		100	0.912	0.926	0.989	0.908	0.921	0.989	0.948	0.906	0.975
		200	0.934	0.922	0.995	0.936	0.943	0.994	0.940	0.893	0.999
		500	0.949	0.946	0.998	0.946	0.952	0.998	0.947	0.904	0.998
	0.5	50	0.865	0.857	0.964	0.880	0.874	0.967	0.938	0.930	0.986
		100	0.907	0.915	0.984	0.910	0.917	0.985	0.937	0.933	0.986
		200	0.933	0.935	0.993	0.930	0.934	0.992	0.937	0.940	0.993
		500	0.946	0.949	0.997	0.945	0.938	0.997	0.938	0.942	0.995
	0.9	50	0.919	0.722	0.945	0.898	0.897	0.916	0.913	0.921	0.973
		100	0.957	0.782	0.970	0.923	0.901	0.995	0.937	0.911	0.975
		200	0.846	0.845	0.901	0.984	0.912	0.995	0.960	0.949	0.991
		500	0.906	0.912	0.966	0.993	0.938	0.993	0.984	0.961	0.995
Contaminated Normal	θ	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	0.2	50	6.330	6.348	0.984	6.918	6.998	0.992	5.808	5.907	0.973
		100	7.023	7.029	0.988	8.909	8.960	0.993	7.563	7.634	0.985
		200	7.043	7.045	0.992	9.006	9.028	0.994	8.278	8.313	0.994
		500	7.042	7.047	0.998	9.040	9.050	0.983	8.595	8.607	0.998
	0.5	50	6.249	6.424	0.979	6.282	6.422	0.972	5.617	5.946	0.962
		100	6.612	6.677	0.979	7.887	8.092	0.974	7.127	7.454	0.965
		200	7.070	7.120	0.991	8.559	8.650	0.993	7.928	8.117	0.985
		500	7.300	7.316	0.996	8.961	9.040	0.998	8.306	8.405	0.976
	0.9	50	4.688	4.783	0.972	3.602	3.898	0.917	2.808	3.048	0.902
		100	5.627	5.778	0.975	4.058	4.802	0.911	3.542	4.248	0.926
		200	5.643	5.768	0.975	5.286	5.524	0.915	6.736	7.092	0.931
		500	6.377	6.380	0.979	7.013	7.059	0.943	9.868	9.989	0.947
Skewed Contaminated Normal	θ	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	0.2	50	11.566	11.568	0.974	10.006	10.152	0.883	7.278	7.474	0.854
		100	12.411	12.489	0.981	12.914	13.107	0.929	10.030	10.296	0.876
		200	12.413	12.441	0.980	14.171	14.356	0.913	11.797	12.080	0.867
		500	12.932	12.957	0.991	14.933	15.063	0.904	13.038	13.341	0.871
	0.5	50	10.258	10.529	0.941	8.992	9.920	0.929	6.265	7.020	0.870
		100	11.451	11.821	0.951	10.705	11.580	0.847	8.261	9.527	0.870
		200	12.131	12.323	0.944	13.589	14.261	0.916	10.858	12.003	0.863
		500	12.831	12.955	0.953	14.437	14.961	0.915	12.440	13.493	0.872
	0.9	50	7.358	7.768	0.895	3.803	4.534	0.669	2.717	3.208	0.666
		100	8.337	8.879	0.869	5.465	7.107	0.715	4.342	5.964	0.684
		200	10.881	11.117	0.871	9.620	11.528	0.744	7.173	9.421	0.698
		500	11.947	12.002	0.869	13.599	14.240	0.787	10.944	12.078	0.688

Table 2: Summary of empirical ARE results of the MA (2) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels for the procedures MIS1, MIS2, and IS. The tabled entries are the ratios of MSE for the MLE estimates divided by the MSE of the procedure.

Normal	(θ_1,θ_2)	σ	0.5			1			2		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.1,0.2)	50	0.930	0.867	0.991	0.920	0.840	0.991	0.917	0.847	0.981
		100	0.916	0.898	0.990	0.932	0.900	0.986	0.920	0.912	0.990
		200	0.927	0.916	0.985	0.902	0.895	0.961	0.919	0.923	0.984
		500	0.918	0.919	0.970	0.902	0.887	0.966	0.930	0.923	0.983
	(0.3,0.4)	50	0.903	0.877	0.999	0.907	0.880	0.991	0.908	0.877	0.991
		100	0.924	0.921	0.998	0.924	0.913	0.990	0.913	0.909	0.988
		200	0.923	0.916	0.977	0.918	0.927	0.982	0.931	0.919	0.992
		500	0.924	0.924	0.974	0.908	0.909	0.963	0.920	0.919	0.982
	(0.8,0.1)	50	0.921	0.898	0.994	0.905	0.904	0.995	0.928	0.889	0.995
		100	0.924	0.925	0.992	0.930	0.921	0.999	0.921	0.925	0.987
		200	0.933	0.937	0.994	0.920	0.917	0.986	0.922	0.919	0.979
		500	0.940	0.940	0.995	0.917	0.917	0.973	0.925	0.930	0.981
Contaminated Normal	(θ_1,θ_2)	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.1,0.2)	50	5.578	5.580	0.976	5.588	5.714	0.965	4.505	5.378	0.985
		100	6.529	6.588	0.982	7.381	7.828	0.972	6.178	6.671	0.959
		200	6.734	6.758	0.987	8.311	8.392	0.982	7.143	7.453	0.935
		500	6.918	6.934	0.970	8.752	8.871	0.985	7.959	8.075	0.939
	(0.3,0.4)	50	5.884	5.990	0.961	6.333	6.869	0.959	4.866	5.260	0.978
		100	6.579	6.688	0.993	7.691	8.128	0.982	6.737	7.009	0.995
		200	6.841	6.891	0.990	8.525	8.669	0.991	7.497	7.780	0.958
		500	7.071	7.117	0.995	8.867	8.951	0.989	8.079	8.161	0.960
	(0.8,0.1)	50	6.444	6.454	0.960	6.760	6.836	0.967	5.614	5.728	0.986
		100	6.843	6.912	0.989	8.346	8.454	0.975	7.186	7.305	0.995
		200	6.852	6.916	0.976	8.718	8.732	0.983	7.976	8.055	0.993
		500	6.835	6.843	0.953	9.023	9.040	1.008	8.574	8.602	0.996
Skewed Contaminated Normal	(θ_1,θ_2)	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.1,0.2)	50	8.791	8.878	0.945	7.022	7.180	0.917	4.753	5.674	0.878
		100	10.695	10.748	0.925	10.528	11.166	0.916	7.888	8.517	0.874
		200	12.036	12.123	0.950	12.606	12.729	0.902	9.968	10.401	0.866
		500	12.191	12.218	0.933	13.794	13.981	0.882	11.570	11.738	0.874
	(0.3,0.4)	50	9.647	9.821	0.940	7.532	8.169	0.902	5.152	5.569	0.867
		100	11.259	11.446	0.940	11.176	11.812	0.907	8.055	8.379	0.867
		200	11.987	12.073	0.940	12.688	12.903	0.899	9.881	10.254	0.866
		500	12.135	12.215	0.930	13.984	14.117	0.894	11.190	11.304	0.860
	(0.8,0.1)	50	10.602	10.617	0.944	9.558	9.666	0.912	6.587	6.720	0.882
		100	11.868	11.988	0.944	12.314	12.474	0.904	9.352	9.507	0.875
		200	12.368	12.484	0.940	13.852	13.874	0.905	11.368	11.480	0.878
		500	12.472	12.487	0.932	14.640	14.668	0.896	12.278	12.317	0.856

Table 3: Summary of empirical ARE results of the MA(3) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels for the procedures MIS1, MIS2, and IS. The tabled entries are the ratios of MSE for the MLE estimates divided by the MSE of the procedure

Normal	$(\theta_1, \theta_2, \theta_3)$	σ	0.5			1			2		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0, 0.0, 0.2)	50	0.910	0.860	0.938	0.913	0.897	0.987	0.936	0.851	0.982
		100	0.998	0.940	0.994	0.925	0.921	0.994	0.940	0.902	0.985
		200	0.988	0.930	0.986	0.928	0.912	0.983	0.945	0.927	0.983
		500	0.976	0.913	0.971	0.945	0.949	1.000	0.919	0.918	0.973
	(0.0, 0.0, 0.5)	50	0.903	0.905	0.967	0.911	0.865	0.987	0.930	0.933	0.992
		100	0.925	0.898	0.974	0.914	0.902	0.981	0.939	0.901	0.984
		200	0.911	0.892	0.964	0.922	0.908	0.973	0.934	0.910	0.967
		500	0.925	0.923	0.979	0.912	0.909	0.972	0.905	0.878	0.927
	(0.0, 0.0, 0.9)	50	0.917	0.915	0.945	0.937	0.910	0.954	0.963	0.958	0.973
		100	0.915	0.926	0.955	0.906	0.896	0.929	0.946	0.940	0.954
		200	0.925	0.905	0.949	0.903	0.904	0.904	0.942	0.939	0.957
		500	0.922	0.872	0.937	0.923	0.915	0.973	0.931	0.921	0.932

Table 3: Continue

Contaminated Normal	(θ_1, θ_2, θ_3)	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.2)	50	6.639	6.846	0.991	6.543	6.834	0.957	5.465	5.655	0.969
		100	6.627	6.699	0.960	7.881	8.004	0.977	6.713	6.833	0.964
		200	6.796	6.811	0.965	8.631	8.727	0.986	7.915	8.037	0.975
		500	6.766	6.783	0.951	8.900	8.945	0.995	8.516	8.542	0.983
	(0.0,0.0,0.5)	50	5.334	5.946	0.989	4.753	6.340	0.963	3.567	4.769	0.972
		100	6.246	6.463	0.986	6.775	7.744	0.966	5.756	6.576	0.974
		200	6.798	6.962	0.985	8.052	8.455	0.978	7.300	7.897	0.984
		500	6.960	7.042	0.974	8.687	8.852	0.984	7.899	8.131	0.959
	(0.0,0.0,0.9)	50	3.281	3.482	0.874	2.415	2.955	0.865	1.689	2.041	0.833
		100	3.570	4.481	0.903	2.979	4.014	0.857	2.105	2.766	0.842
		200	4.320	4.694	0.903	3.873	4.606	0.893	3.038	4.531	0.894
		500	5.979	6.003	0.939	6.532	7.055	0.939	6.051	6.285	0.939
Skewed Contaminated Normal	(θ_1, θ_2, θ_3)	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.2)	50	10.262	10.546	0.911	8.810	9.313	0.924	6.007	6.510	0.893
		100	11.540	11.738	0.923	12.101	12.724	0.903	9.237	9.903	0.893
		200	12.281	12.482	0.945	13.508	14.026	0.890	10.846	11.489	0.870
		500	12.105	12.207	0.925	13.927	14.200	0.848	12.143	12.670	0.868
	(0.0,0.0,0.5)	50	8.201	9.672	0.937	5.406	7.363	0.880	3.640	4.978	0.876
		100	10.223	11.208	0.928	9.480	13.178	0.896	6.353	8.670	0.881
		200	11.303	12.102	0.910	11.691	13.267	0.907	8.621	10.871	0.876
		500	11.762	12.041	0.930	13.262	14.416	0.913	9.827	11.630	0.849
	(0.0,0.0,0.9)	50	4.138	4.615	0.812	2.462	2.711	0.762	1.567	1.623	0.692
		100	5.039	6.224	0.818	3.135	4.920	0.767	1.815	2.550	0.703
		200	6.412	6.891	0.859	4.132	6.942	0.761	2.477	3.389	0.728
		500	9.288	9.507	0.878	8.627	10.028	0.828	6.241	6.986	0.815

Table 4: Summary of empirical ARE results of the MA(4) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels for the procedures MIS1, MIS2, and IS. The tabled entries are the ratios of MSE for the MLE estimates divided by the MSE of the procedure

		σ	0.5	1	2						
Normal	(θ_1, θ_2, θ_3, θ_4)	n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.0,0.2)	50	0.919	0.876	0.954	0.913	0.889	0.979	0.932	0.911	0.972
		100	0.977	0.924	0.992	0.923	0.942	0.974	0.930	0.921	0.981
		200	0.995	0.912	0.980	0.908	0.924	0.963	0.924	0.932	0.990
		500	0.982	0.915	0.963	0.911	0.922	0.962	0.927	0.954	0.977
	(0.0,0.0,0.0,0.5)	50	0.905	0.845	0.988	0.916	0.910	0.982	0.955	0.841	0.979
		100	0.910	0.913	0.977	0.928	0.912	0.975	0.975	0.903	0.982
		200	0.942	0.907	0.957	0.937	0.927	0.943	0.954	0.947	0.955
		500	0.918	0.910	0.913	0.906	0.902	0.904	0.913	0.900	0.915
	(0.0,0.0,0.0,0.9)	50	0.932	0.909	0.952	0.919	0.918	0.962	0.908	0.908	0.988
		100	0.913	0.906	0.913	0.944	0.934	0.944	0.965	0.949	0.977
		200	0.931	0.910	0.918	0.937	0.919	0.937	0.951	0.928	0.955
		500	0.936	0.903	0.984	0.915	0.908	0.928	0.950	0.935	0.960
Contaminated Normal	(θ_1, θ_2, θ_3, θ_4)	ϵ	10%	20%	30%						
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.0,0.2)	50	6.836	7.093	0.988	6.262	6.725	0.988	4.953	5.413	0.979
		100	6.805	6.931	0.993	7.870	8.104	0.978	6.877	7.198	0.992
		200	7.009	7.018	0.981	8.563	8.675	0.993	7.858	8.035	0.980
		500	6.847	6.871	0.956	8.779	8.800	0.960	8.147	8.216	0.959
	(0.0,0.0,0.0,0.5)	50	5.011	6.271	0.992	4.254	5.706	0.987	3.176	4.257	0.983
		100	6.244	6.622	0.996	6.263	7.457	0.962	5.114	6.265	0.964
		200	6.644	6.874	0.982	7.835	8.350	0.976	6.662	7.404	0.950
		500	6.832	6.881	0.987	7.651	7.904	0.882	7.184	7.428	0.899
	(0.0,0.0,0.0,0.9)	50	2.905	3.058	0.879	2.263	2.287	0.868	1.492	1.644	0.804
		100	3.275	4.196	0.880	2.479	3.266	0.833	1.817	2.475	0.821
		200	3.551	4.242	0.815	3.177	4.800	0.846	2.381	4.003	0.833
		500	5.189	5.454	0.858	5.373	5.750	0.836	4.765	5.097	0.892

Table 4: Continue

Skewed Contaminated Normal	$(\theta_1, \theta_2, \theta_3, \theta_4)$	ϵ	10%			20%			30%		
		n	MIS1	MIS2	IS	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.0,0.2)	50	10.488	10.809	0.975	8.974	9.874	0.934	5.288	6.198	0.910
		100	11.760	12.027	0.941	11.983	12.628	0.914	8.812	9.711	0.904
		200	12.327	12.581	0.943	13.639	14.220	0.925	10.989	12.021	0.881
		500	12.318	12.396	0.937	14.118	14.494	0.895	12.261	12.820	0.895
	(0.0,0.0,0.0,0.5)	50	6.830	8.572	0.918	4.462	6.622	0.900	3.048	4.442	0.885
		100	9.522	11.060	0.920	7.985	11.292	0.900	5.193	8.623	0.894
		200	10.739	11.695	0.912	10.986	13.064	0.893	7.533	10.247	0.850
		500	11.211	11.620	0.893	11.719	13.339	0.814	8.769	10.980	0.775
	(0.0,0.0,0.0,0.9)	50	4.109	5.009	0.823	2.095	2.269	0.737	1.364	1.446	0.679
		100	4.420	5.553	0.793	2.655	3.653	0.741	1.568	1.861	0.665
		200	5.028	6.144	0.797	3.144	5.304	0.737	2.027	3.162	0.722
		500	7.947	8.529	0.827	6.492	8.976	0.810	4.133	4.517	0.774

Table 5: Summary of empirical ARE results of the MIS1, MIS2, and IS procedures for the MA(1), MA(2), MA(3), and MA(4) models simulated from the normal additive outlier models (NAO) and the skewed contaminated normal additive outlier models (SCNAO). The tabled entries are the ratios of MSE for the MLE estimates divided by the MSE of the procedure

			Case 4			Case 5		
MA(1)	θ	n	MIS1	MIS2	IS	MIS1	MIS2	IS
	0.2	50	15.629	15.975	0.974	12.369	12.973	0.976
		100	36.786	38.666	0.989	26.628	42.772	0.989
		200	77.906	79.228	0.996	54.136	66.553	0.995
		500	193.310	198.952	0.998	138.820	156.018	0.998
	0.5	50	13.652	18.398	0.964	9.804	11.611	0.962
		100	29.226	38.122	0.986	20.913	23.611	0.985
		200	70.121	100.351	0.994	43.846	53.446	0.994
		500	165.665	210.272	0.997	110.100	133.003	0.997
	0.9	50	4.850	5.098	0.827	3.604	4.010	0.820
		100	9.655	11.267	0.871	7.301	8.205	0.880
		200	16.865	18.493	0.895	12.549	16.882	0.907
		500	39.341	49.168	0.968	22.610	26.663	0.972
			Case 4			Case 5		
MA(2)	(θ_1, θ_2)	n	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.1,0.2)	50	14.981	15.826	0.996	11.454	11.802	0.990
		100	33.115	34.376	0.986	24.385	24.633	0.996
		200	70.591	72.294	0.985	50.112	61.603	0.983
		500	166.252	178.570	0.950	122.129	162.168	0.968
	(0.3,0.4)	50	10.891	13.158	0.981	8.891	11.123	0.984
		100	26.907	33.896	0.978	19.716	24.187	0.993
		200	62.462	76.064	0.977	42.741	50.291	0.991
		500	142.273	179.945	0.991	102.528	124.232	0.964
	(0.8,0.1)	50	8.911	13.251	0.994	7.409	10.620	0.991
		100	21.469	32.578	0.997	16.865	23.351	0.986
		200	47.056	69.417	0.978	37.879	50.977	0.982
		500	121.575	170.750	0.962	91.068	122.350	0.963
			Case 4			Case 5		
MA(3)	$(\theta_1, \theta_2, \theta_3)$	n	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.2)	50	13.097	13.662	0.981	9.579	14.612	0.983
		100	31.381	33.261	0.984	23.046	39.324	0.983
		200	67.346	69.671	0.960	48.370	75.540	0.964
		500	165.301	181.922	0.954	116.997	124.190	0.949
	(0.0,0.0,0.5)	50	7.769	11.387	0.979	5.803	8.773	0.984
		100	20.249	32.389	0.965	13.539	20.403	0.971
		200	41.537	68.376	0.970	32.625	48.875	0.967
		500	109.359	186.854	0.956	77.840	124.965	0.938
	(0.0,0.0,0.9)	50	3.251	4.208	0.850	2.237	2.269	0.833
		100	5.019	6.545	0.853	3.162	5.024	0.864

Table 5: Continue

		200	7.679	11.081	0.883	5.091	11.433	0.866
		500	25.702	55.684	0.920	18.513	29.791	0.916
			Case 4			Case 5		
MA(4)	(θ_1, θ_2, θ_3, θ_4)	n	MIS1	MIS2	IS	MIS1	MIS2	IS
	(0.0,0.0,0.0,0.2)	50	7.813	8.15800	0.892	7.648	8.421	0.907
		100	46.290	66.09700	0.987	30.883	47.206	0.968
		200	63.772	93.52800	0.977	42.660	89.679	0.957
		500	142.361	156.74300	0.982	131.691	167.567	0.989
	(0.0,0.0,0.0,0.5)	50	9.823	13.99800	0.959	5.063	9.418	0.837
		100	15.736	32.20200	0.983	11.947	21.433	0.977
		200	29.422	53.07600	0.999	24.976	39.677	0.997
		500	76.436	137.43300	0.903	71.199	130.185	0.952
	(0.0,0.0,0.0,0.9)	50	4.211	4.37800	0.930	2.941	3.971	0.878
		100	5.128	9.02913	0.833	4.683	8.007	0.839
		200	6.433	12.38500	0.821	3.601	10.196	0.824
		500	18.969	22.90700	0.783	14.126	20.151	0.854

Monte Carlo Study of Forecasting by the Estimators

Setup of the Simulation

In our second empirical investigation, we compared the forecasting abilities of the four procedures. For each procedure, we computed one-step ahead forecasts. For example, for $n = 50$, for each iteration of the simulation of a situation, 51 observations were generated from the model. The observation y_{51} was considered as the true value and the model was fit using only the first 50 values. Then the forecast of y_{51} is the predicted value y_b51 based on the fit. The empirical bias is the difference$y_{51} - \widehat{y}_{51}$. These biases were recorded from which Mean Square Errors (MSE) were computed.

We ran this forecasting simulation for all four estimates: MIS1, MIS2, IS, and MLE. The simulation size for each situation is 1000. The MSEs for all four estimating procedures are recorded in Tables 6-10.

Summary of the Results of the Simulation

Tables 6-10 present the results of the forecasting simulation. Each table displays, over its situations, the MSE of the estimator's (MIS1, MIS2, IS, or MLE) predicted values from the "true" values. We briefly discuss these results.

The top panel of Table 6 displays the MSEs for the normal situations of the MA(1) models. Overall, the MLE's prediction performs best in terms of MSE; but, in 6 of these situations, the MIS2 prediction has lower MSE than the MLE prediction. In the other situations, the MSEs of the MIS2 procedure are quite close to those of the MLE procedure. The MSEs for the IS and MLE procedures are quite close, but in every normal situation, the MLE's predictions outperform those of the IS predictions. For the two robust procedures, in every situation, the MSEs of the MIS2 predictions are lower than those of the MIS1 predictions. The results are the same for the MA(2), MA(3), and MA(4) normal situations found in the top panels of the respective Tables 7-9. The MLE and MIS2 procedures outperform the IS and MIS1 procedures, respectively, while the predictions of the MLE procedure outperform the predictions of the MIS2 procedure. Note that the MSEs tend to increase as σ increases and as the MA coefficients increase.

The bottom two panels of Tables 6-10 contain the results for the innovative outlier situations for the MA(1)-MA(4) models, respectively. The results for the symmetric contaminated normal situations are found in the middle panel. The MIS2 and MIS1 predictions are much more precise than those of the IS and MLE procedures for each of these situations. The ratio of the MSEs, MLE to MIS2, ranges from 5 to 10 across all MA models. Of the two, the predictions of the MIS2 procedure are slightly more precise than the MIS1 procedure. Although both the MLE and IS forecasts are poor in terms of efficiency in these situations, the MLE predictions are more precise than those of the IS procedure. The MSEs for all methods tend to increase with increasing contamination.

The results are similar for the skewed contaminated normal situations in the bottom panel of these tables. The predictions of the MIS2 and MIS1 procedures outperform those of the IS and MLE procedures with similar ranges of precision. In terms of precision, the best predictions are by the MIS2 method but the MIS1 is only slightly less precise.

The results for the additive outlier situations are in Table 10 under Cases 4 and 5 for the symmetric AO and skewed AO situations, respectively. The dominance of the MIS procedures is evident. In many of these situations, the MLE predictions are 100 to 200 times less precise than those of the MIS methods. Of the two MIS procedures, MIS1 is slightly less precise than MIS2 for all situations.

Table 6: Summary of empirical one-step ahead forecasts MSE results for the MIS1, MIS2, IS, and MLE procedures for the MA(1) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels. The tabled entries are the MSEs of forecasts from the true value for each procedure.

Normal	θ	σ	0.5				1				2			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	0.2	50	0.1062	0.1059	0.1057	0.1020	0.2117	0.2080	0.2082	0.1981	0.3528	0.3492	0.3501	0.3306
		100	0.1027	0.1026	0.1013	0.0951	0.2116	0.2113	0.2116	0.2040	0.3381	0.3343	0.3347	0.3280
		200	0.0987	0.0986	0.0984	0.0980	0.2004	0.2001	0.2001	0.1999	0.3300	0.3259	0.3280	0.3254
		500	0.0982	0.0982	0.0980	0.0977	0.1998	0.1983	0.1984	0.1883	0.3266	0.2241	0.3247	0.3238
	0.5	50	0.1170	0.1157	0.1164	0.1161	0.2377	0.2372	0.2376	0.2373	0.3918	0.3874	0.3915	0.3871
		100	0.1161	0.1153	0.1158	0.1158	0.2355	0.2341	0.2348	0.2347	0.3914	0.3872	0.3907	0.3864
		200	0.1156	0.1151	0.1156	0.1155	0.2340	0.2339	0.2325	0.2322	0.3892	0.3866	0.3869	0.3861
		500	0.1152	0.1150	0.1149	0.1141	0.2335	0.2327	0.2331	0.2301	0.3887	0.3861	0.3866	0.3859
	0.9	50	0.1287	0.1271	0.1284	0.1261	0.2617	0.2606	0.2604	0.2592	0.4347	0.4342	0.4344	0.4308
		100	0.1293	0.1263	0.1288	0.1244	0.2604	0.2595	0.2601	0.2591	0.4316	0.4308	0.4309	0.4301
		200	0.1250	0.1248	0.1231	0.1213	0.2591	0.2590	0.2589	0.2589	0.4245	0.4214	0.4237	0.4204
		500	0.1238	0.1235	0.1225	0.1206	0.2568	0.2535	0.2558	0.2522	0.4064	0.4057	0.4060	0.4053
Contaminated Normal	θ	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	0.2	50	0.4016	0.3816	2.0372	2.0359	0.4990	0.4641	3.0322	3.0283	0.6143	0.5713	3.9411	3.8443
		100	0.3861	0.3475	1.9909	1.9893	0.4591	0.4545	2.7699	2.7688	0.6102	0.6041	3.7930	3.7279
		200	0.3727	0.3466	1.7757	1.7699	0.4428	0.4207	2.6964	2.6904	0.5974	0.5675	3.6037	3.6003
		500	0.3260	0.3227	1.7581	1.7411	0.4041	0.3637	2.6181	2.5011	0.5607	0.5047	3.5552	3.3941
	0.5	50	0.4260	0.4132	2.2272	2.2158	0.5808	0.5518	3.5055	3.5045	0.7159	0.6801	4.3666	4.3518
		100	0.4091	0.3887	2.1503	2.1496	0.5634	0.5070	3.4625	3.4543	0.6937	0.6244	4.3487	4.3260
		200	0.3761	0.3385	2.1260	2.1127	0.5153	0.4792	3.1604	3.1483	0.6936	0.6451	4.1843	4.1771
		500	0.3374	0.3138	1.9427	1.9401	0.5142	0.5090	3.1075	3.1015	0.6631	0.6565	4.0590	4.0338
	0.9	50	0.4495	0.4450	2.7313	2.6202	0.6940	0.6870	4.1979	3.8952	0.8624	0.8538	5.2744	5.1447
		100	0.4292	0.4163	2.6951	2.5471	0.6512	0.6186	3.9529	3.7656	0.8562	0.8134	5.2702	4.9736
		200	0.4194	0.3984	2.6546	2.5451	0.6115	0.5504	3.6996	3.6301	0.8523	0.7671	5.2262	4.9226
		500	0.3138	0.2824	2.3465	2.2631	0.6022	0.5961	3.6966	3.4819	0.8180	0.8098	4.9653	4.6737
Skewed Contaminated Normal	θ	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	0.2	50	0.4016	0.3816	4.0781	4.0646	0.6210	0.6024	6.4152	6.3611	0.7999	0.7599	8.3630	8.2182
		100	0.3861	0.3475	3.8950	3.8889	0.6119	0.5813	6.2325	6.1898	0.7927	0.7134	8.3530	8.1599
		200	0.3727	0.3466	3.7714	3.7586	0.5817	0.5235	6.0415	5.9624	0.7926	0.7371	8.2578	8.1364
		500	0.3260	0.3227	3.3559	3.3180	0.5501	0.5116	5.6849	5.6069	0.7696	0.7619	8.2503	8.0946
	0.5	50	0.5112	0.4959	5.1250	5.1246	0.7667	0.7284	7.9811	7.7851	1.1552	1.1206	12.1565	11.7961
		100	0.4909	0.4664	4.9719	4.9354	0.7220	0.6498	7.4909	7.3357	0.9786	0.9297	10.5265	10.0402
		200	0.4513	0.4062	4.6499	4.5812	0.6750	0.6278	6.9903	6.7935	0.9676	0.8708	10.3100	9.7772
		500	0.4048	0.3765	4.1885	4.1026	0.5745	0.5688	6.0392	5.8533	0.9115	0.8477	9.7485	9.2691
	0.9	50	0.6068	0.6008	6.8533	6.4145	0.9016	0.8746	11.6518	10.2673	1.3118	1.2986	13.8544	12.8007
		100	0.5794	0.5620	6.3853	6.3446	0.8703	0.8268	9.0989	8.6262	1.3016	1.2625	13.3167	11.4638
		200	0.5662	0.5379	5.7989	5.6940	0.7535	0.6781	8.6009	8.0482	1.2322	1.1706	12.8569	11.0932
		500	0.4236	0.3812	4.3304	4.1955	0.7339	0.6826	7.6773	7.2111	1.1472	1.0324	11.5865	10.8332

Table 7: Summary of empirical one-step ahead forecasts MSE results for the MIS1, MIS2, IS, and MLE procedures for the MA(2) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels. The tabled entries are the MSEs of forecasts from the true value for each procedure.

	(θ_1, θ_2)	σ	0.5				1				2			
Normal		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.1,0.2)	50	0.0791	0.0762	0.0755	0.0705	0.3579	0.3031	0.3462	0.2806	0.5529	0.4872	0.4825	0.4511
		100	0.0770	0.0742	0.0740	0.0700	0.3299	0.2944	0.3168	0.2777	0.5027	0.4709	0.4596	0.4442
		200	0.0760	0.0717	0.0726	0.0689	0.3204	0.2877	0.2978	0.2766	0.4789	0.4592	0.4504	0.4416
		500	0.0746	0.0701	0.0720	0.0688	0.3170	0.2796	0.2917	0.2741	0.4676	0.4348	0.4440	0.4263
	(0.3,0.4)	50	0.0681	0.0649	0.0676	0.0607	0.2888	0.2586	0.2751	0.2417	0.4389	0.4092	0.4290	0.3824
		100	0.0673	0.0636	0.0669	0.0606	0.2728	0.2537	0.2668	0.2416	0.4263	0.4075	0.3999	0.3881
		200	0.0655	0.0625	0.0649	0.0601	0.2692	0.2495	0.2588	0.2399	0.4169	0.3746	0.3899	0.3602
		500	0.0653	0.0609	0.0646	0.0594	0.2602	0.2456	0.2572	0.2396	0.4132	0.3624	0.3914	0.3535
	(0.8,0.1)	50	0.0579	0.0573	0.0570	0.0496	0.2318	0.2134	0.2261	0.2013	0.3660	0.3413	0.3596	0.3166
		100	0.0563	0.0528	0.0552	0.0503	0.2250	0.2113	0.2197	0.2012	0.3538	0.3380	0.3438	0.3219
		200	0.0557	0.0517	0.0541	0.0502	0.2227	0.2060	0.2172	0.2000	0.3496	0.3324	0.3446	0.3227
		500	0.0549	0.0508	0.0538	0.0498	0.2193	0.2003	0.2145	0.1964	0.3432	0.3287	0.3363	0.3222
Contaminated Normal	(θ_1, θ_2)	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.1,0.2)	50	0.4670	0.4437	2.9766	2.7700	0.8123	0.7879	5.5558	5.1152	1.5031	1.4881	8.2532	7.4353
		100	0.4369	0.3932	2.9065	2.5955	0.7009	0.6658	5.5323	4.8878	1.1588	1.1240	8.1497	7.0454
		200	0.4180	0.3887	2.8969	2.5597	0.6439	0.5795	5.5114	4.8180	1.0155	0.9647	8.0560	6.9914
		500	0.4099	0.3853	2.8000	2.5530	0.6136	0.5706	5.3274	4.8059	0.9572	0.8615	7.9205	6.8911
	(0.3,0.4)	50	0.4442	0.4398	2.6326	2.4499	0.8548	0.8121	4.9698	4.5757	1.5809	1.5335	7.3961	6.6631
		100	0.3947	0.3907	2.6266	2.3209	0.6539	0.5885	4.9370	4.4018	1.0960	1.0412	7.3532	6.3252
		200	0.3793	0.3679	2.5216	2.2755	0.5816	0.5409	4.9303	4.3159	0.9438	0.8494	7.2404	6.2899
		500	0.3677	0.3309	2.5072	2.2153	0.5479	0.5424	4.8464	4.2368	0.8663	0.8057	7.1771	6.1393
	(0.8,0.1)	50	0.3855	0.3469	2.1938	2.0622	0.7160	0.6945	4.1992	3.9052	1.4367	1.3649	6.2085	5.6497
		100	0.3372	0.3203	2.1853	1.9383	0.5631	0.5350	4.1537	3.6124	0.9712	0.8741	6.1501	5.2387
		200	0.3182	0.2863	2.1706	1.9104	0.5004	0.4504	4.0887	3.5925	0.8194	0.7620	6.0598	5.2048
		500	0.3049	0.2836	2.1562	1.8444	0.4641	0.4316	3.9593	3.3508	0.7272	0.7199	6.0310	4.9943
Skewed Contaminated Normal	(θ_1, θ_2)	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.1,0.2)	50	0.5222	0.5065	5.4759	4.8380	1.1006	0.9906	9.9853	9.1046	2.2613	2.2387	14.2204	12.7402

Table 7: Continue

		100	0.4583	0.4354	5.3481	4.8246	0.8291	0.8208	9.8656	8.9200	1.5102	1.4347	14.0798	12.6903
		200	0.4352	0.3917	5.3159	4.7655	0.7176	0.6961	9.8194	8.7688	1.2468	1.1221	13.9879	12.1762
		500	0.4219	0.3923	5.1759	4.6471	0.6701	0.6634	9.5852	8.7172	1.1218	1.1106	13.8505	12.1053
	(0.3,0.4)	50	0.5166	0.4805	4.7469	4.3482	1.1901	1.1782	8.8386	8.1940	2.5787	2.3982	12.5933	11.2207
		100	0.4255	0.4212	4.7408	4.1768	0.8138	0.7894	8.8092	7.8707	1.5673	1.5517	12.4615	11.0033
		200	0.3932	0.3892	4.7227	4.1498	0.6834	0.6151	8.8051	7.8098	1.2778	1.2650	12.4584	10.8907
		500	0.3796	0.3758	4.6783	4.1444	0.6181	0.5871	8.7220	7.7838	1.0927	1.0818	12.3196	10.7767
	(0.8,0.1)	50	0.4583	0.4446	3.9885	3.5072	1.1528	1.0952	7.4198	6.9746	2.1816	2.0289	10.4796	9.4536
		100	0.3670	0.3303	3.9617	3.4921	0.7200	0.6480	7.4020	6.6100	1.3664	1.2298	10.4684	9.1592
		200	0.3322	0.3156	3.9526	3.4264	0.5674	0.5277	7.3670	6.4402	1.0467	0.9943	10.4230	8.9568
		500	0.3174	0.2857	3.7712	3.3755	0.5135	0.5084	7.3650	6.3000	0.8773	0.7896	10.2757	8.7262

Table 8: Summary of empirical one-step ahead forecasts MSE results for the MIS1, MIS2, IS, and MLE procedures for the MA(3) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels. The tabled entries are the MSEs of forecasts from the true value for each procedure

		σ	0.5				1				2			
Normal	(θ_1, θ_2, θ_3)	n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.2)	50	0.0547	0.0509	0.0498	0.0453	0.2182	0.2073	0.1989	0.1830	0.3448	0.3413	0.3181	0.2990
		100	0.0531	0.0526	0.0494	0.0449	0.2142	0.1927	0.1978	0.1820	0.3432	0.3260	0.3152	0.2963
		200	0.0530	0.0504	0.0481	0.0438	0.2121	0.2015	0.1942	0.1786	0.3392	0.3053	0.3109	0.2922
		500	0.0526	0.0474	0.0465	0.0423	0.2103	0.1893	0.1860	0.1711	0.3370	0.3202	0.2959	0.2781
	(0.0,0.0,0.5)	50	0.0662	0.0629	0.0600	0.0546	0.2659	0.2473	0.2387	0.2196	0.4268	0.3841	0.3810	0.3581
		100	0.0642	0.0578	0.0594	0.0541	0.2557	0.2429	0.2378	0.2188	0.4118	0.3912	0.3774	0.3548
		200	0.0635	0.0590	0.0588	0.0535	0.2547	0.2292	0.2337	0.2150	0.4025	0.3623	0.3757	0.3531
		500	0.0633	0.0626	0.0569	0.0518	0.2516	0.2340	0.2258	0.2077	0.4024	0.3743	0.3649	0.3430
	(0.0,0.0,0.9)	50	0.0913	0.0904	0.0727	0.0662	0.3625	0.3588	0.2847	0.2619	0.5323	0.5269	0.4621	0.4344
		100	0.0863	0.0820	0.0703	0.0640	0.3403	0.3233	0.2832	0.2605	0.5254	0.5202	0.4556	0.4283
		200	0.0792	0.0713	0.0697	0.0635	0.3136	0.2822	0.2775	0.2553	0.5047	0.4996	0.4426	0.4160
		500	0.0725	0.0717	0.0684	0.0623	0.2910	0.2735	0.2745	0.2525	0.4624	0.4578	0.4377	0.4114
Contaminated Normal	(θ_1,θ_2,θ_3)	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.2)	50	0.3615	0.3579	2.1723	2.0202	0.6445	0.6123	3.7965	3.5801	1.2223	1.1612	5.7872	5.2663
		100	0.3225	0.3129	2.1233	1.9747	0.5297	0.4768	4.0275	3.7980	0.8821	0.7939	5.8729	5.3444
		200	0.3093	0.2938	2.1149	1.9668	0.4842	0.4503	4.1554	3.9185	0.7735	0.7194	6.1106	5.5606
		500	0.3040	0.2736	2.0459	1.9027	0.4528	0.4483	4.1568	3.9198	0.7111	0.7040	6.1161	5.5657
	(0.0,0.0,0.5)	50	0.4976	0.4727	2.6204	2.4370	1.0982	1.0432	4.6923	4.4249	2.0280	1.9266	6.9567	6.3306
		100	0.4222	0.3800	2.6064	2.4240	0.7253	0.6528	4.8211	4.5463	1.2615	1.1353	7.2040	6.5557
		200	0.3848	0.3578	2.5432	2.3652	0.6066	0.5642	4.9948	4.7101	1.0205	0.9490	7.3603	6.6978
		500	0.3692	0.3655	2.4814	2.3077	0.5594	0.5314	4.9900	4.7055	0.8896	0.8451	7.3552	6.6932
	(0.0,0.0,0.9)	50	1.0065	0.9965	3.0533	2.8396	2.5614	2.4845	5.9369	5.5985	4.8535	4.7079	8.6266	7.8502
		100	0.8487	0.1942	3.0262	2.8143	1.9524	1.8547	5.7235	5.3972	4.0222	3.8211	8.6270	7.8506
		200	0.6932	0.1860	3.0220	2.8105	1.5028	1.3525	5.7670	5.4383	2.7527	2.4774	8.5953	7.8217
		500	0.4744	0.2387	2.9861	2.7771	0.8380	0.7794	5.6790	5.3553	1.3538	1.2590	8.4174	7.6599
Skewed Contaminated Normal	(θ_1,θ_2,θ_3)	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.2)	50	0.4019	0.3737	3.9576	3.8389	0.8661	0.8228	7.3701	6.7805	1.7812	1.7634	10.5108	10.0063
		100	0.3397	0.3363	3.9345	3.8164	0.6174	0.5557	7.3482	6.7604	1.1811	1.1220	10.3769	9.8788
		200	0.3198	0.3039	3.9322	3.8142	0.5425	0.5046	7.2674	6.6860	0.9326	0.8393	10.2513	9.7592
		500	0.3102	0.2792	3.8473	3.7319	0.4972	0.4922	7.0395	6.4763	0.8312	0.7897	9.8889	9.4143
	(0.0,0.0,0.5)	50	0.6578	0.6249	4.7938	4.6500	1.2198	1.1832	8.8857	8.1748	3.6577	3.2919	12.5893	11.9850
		100	0.4694	0.4225	4.7754	4.6321	0.7566	0.7188	8.8359	8.1290	2.0049	1.9046	12.5109	11.9104
		200	0.4102	0.3815	4.7396	4.5974	0.7499	0.6749	8.7944	8.0908	1.4296	1.2867	12.3652	11.7717
		500	0.3847	0.3808	4.7132	4.5718	0.6493	0.6038	8.6426	7.9512	1.1732	1.0911	12.3575	11.7643
	(0.0,0.0,0.9)	50	0.5952	0.5892	5.4897	5.3250	4.2239	4.1817	10.6600	9.8072	9.2026	9.1105	15.1402	14.4135
		100	0.5769	0.5480	5.4489	5.2854	3.3566	3.2559	10.2596	9.4388	7.2045	6.9884	14.6762	13.9717
		200	0.5478	0.4930	5.3895	5.2278	2.3415	2.2244	10.1669	9.3535	5.2693	5.0058	14.5163	13.8195
		500	0.5428	0.5373	5.3844	5.2228	1.0871	0.9784	10.0796	9.2732	2.2944	2.0650	14.2195	13.5370

Table 9: Summary of empirical one-step ahead forecasts MSE results for the MIS1, MIS2, IS, and MLE procedures for the MA(4) models simulated from Normal, Contaminated, and Skewed Contaminated Normal distributions according to different variances and contamination levels. The tabled entries are the MSEs of forecasts from the true value for each procedure.

Normal	(θ_1, θ_2, θ_3, θ_4)	σ	0.5				1				2			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.0,0.2)	50	0.0558	0.0541	0.0518	0.0472	0.1934	0.1837	0.1786	0.1643	0.3099	0.2944	0.2846	0.2676
		100	0.0558	0.0530	0.0508	0.0462	0.1914	0.1723	0.1744	0.1605	0.3024	0.2722	0.2790	0.2623
		200	0.0546	0.0491	0.0496	0.0451	0.1879	0.1697	0.1720	0.1582	0.3005	0.2714	0.2754	0.2589
		500	0.0531	0.0480	0.0470	0.0427	0.1827	0.1626	0.1615	0.1486	0.2927	0.2605	0.2570	0.2416
	(0.0,0.0,0.0,0.5)	50	0.0813	0.0772	0.0734	0.0668	0.2799	0.2715	0.2513	0.2312	0.4492	0.4358	0.4032	0.3790
		100	0.0810	0.0729	0.0734	0.0668	0.2778	0.2639	0.2508	0.2308	0.4444	0.4222	0.4030	0.3788
		200	0.0792	0.0707	0.0731	0.0665	0.2733	0.2460	0.2493	0.2294	0.4320	0.3888	0.4010	0.3770
		500	0.0778	0.0669	0.0720	0.0655	0.2665	0.2407	0.2479	0.2281	0.4292	0.3876	0.3934	0.3698
	(0.0,0.0,0.0,0.9)	50	0.1257	0.1219	0.1006	0.0916	0.3993	0.3953	0.3242	0.2982	0.6015	0.5955	0.5216	0.4903
		100	0.1235	0.1173	0.1001	0.0911	0.3896	0.3779	0.3202	0.2945	0.5863	0.5687	0.5105	0.4799
		200	0.1101	0.0991	0.0998	0.0908	0.3488	0.3314	0.3136	0.2885	0.5614	0.5334	0.5091	0.4785

Table 9: Continue

		500	0.1057	0.0983	0.0970	0.0882	0.3394	0.3054	0.3087	0.2840	0.5393	0.4854	0.4923	0.4628
Contaminated Normal	(θ_1, θ_2, θ_3, θ_4)	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.0,0.2)	50	0.3688	0.3503	2.2157	2.0606	0.5654	0.5088	3.3303	3.1404	1.0722	0.9972	5.0765	4.6196
		100	0.3387	0.3048	2.2295	2.0734	0.4783	0.4319	3.6369	3.4296	0.7966	0.7886	5.3032	4.8259
		200	0.3186	0.2963	2.1783	2.0258	0.4289	0.4074	3.6808	3.4710	0.6852	0.6509	5.4128	4.9256
		500	0.3070	0.3040	2.0663	1.9217	0.3933	0.3540	3.6106	3.4048	0.6177	0.5559	5.3125	4.8343
	(0.0,0.0,0.0,0.5)	50	0.6090	0.5786	3.2074	2.9829	1.1560	1.0323	4.9394	4.6578	2.1347	2.0280	7.3229	6.6639
		100	0.5117	0.4605	3.1590	2.9378	0.7560	0.6494	5.0251	4.7387	1.3149	1.1834	7.5089	6.8331
		200	0.4802	0.4466	3.1739	2.9517	0.6511	0.6316	5.3608	5.0552	1.0953	1.0186	7.8996	7.1887
		500	0.4741	0.4504	3.1861	2.9631	0.6177	0.5868	5.5101	5.1960	0.9823	0.9725	8.1219	7.3909
	(0.0,0.0,0.0,0.9)	50	1.3860	1.3444	4.2044	3.9101	2.8216	2.5395	6.5401	6.1673	5.3466	5.2932	9.5031	8.6478
		100	1.2145	1.1538	4.3304	4.0273	2.2351	2.0786	6.5522	6.1787	4.6046	4.3743	9.8762	8.9873
		200	0.9639	0.8675	4.2021	3.9080	1.6717	1.5881	6.4152	6.0496	3.0621	2.7559	9.5614	8.7008
		500	0.6917	0.6433	4.3537	4.0490	0.9775	0.8797	6.6239	6.2464	1.5791	1.5633	9.8181	8.9345
Skewed Contaminated Normal	(θ_1, θ_2, θ_3, θ_4)	ϵ	10%				20%				30%			
		n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.0,0.2)	50	0.4099	0.3812	4.0368	3.9157	0.7598	0.6838	6.4650	5.9478	1.5625	1.4844	9.2201	8.7775
		100	0.3567	0.3531	4.1312	4.0073	0.5575	0.5297	6.6355	6.1046	1.0665	0.9599	9.3703	8.9205
		200	0.3294	0.3130	4.0501	3.9286	0.4806	0.4325	6.4374	5.9224	0.8261	0.7683	9.0806	8.6447
		500	0.3133	0.2820	3.8858	3.7692	0.4319	0.4016	6.1145	5.6253	0.7220	0.7148	8.5895	8.1772
	(0.0,0.0,0.0,0.5)	50	0.8051	0.7649	5.8676	5.6916	1.2840	1.2711	9.3534	8.6052	3.8502	3.6577	13.2520	12.6159
		100	0.5689	0.5121	5.7878	5.6141	0.7886	0.7807	9.2099	8.4731	2.0897	1.8807	13.0404	12.4144
		200	0.5120	0.4761	5.9150	5.7375	0.8049	0.7324	9.4388	8.6837	1.5344	1.4577	13.2713	12.6343
		500	0.4939	0.4890	6.0517	5.8701	0.7170	0.6453	9.5435	8.7800	1.2955	1.1659	13.6456	12.9906
	(0.0,0.0,0.0,0.9)	50	0.8196	0.8114	7.5593	7.3326	4.6531	4.4670	11.7430	10.8036	10.1376	9.4279	16.6784	15.8779
		100	0.8255	0.7842	7.7974	7.5634	3.8427	3.6121	11.7451	10.8055	8.2477	8.1653	16.8013	15.9948
		200	0.7617	0.6856	7.4941	7.2693	2.6046	2.4223	11.3096	10.4049	5.8616	5.8029	16.1479	15.3728
		500	0.7914	0.7835	7.8504	7.6149	1.2680	1.1539	11.7569	10.8163	2.6762	2.4889	16.5856	15.7895

Table 10: Summary of empirical one-step ahead forecasts MSE results for the MIS1, MIS2, IS, and MLE procedures for the MA(1)-MA(4) models simulated from the normal additive outlier models (NAO) and the skewed contaminated normal additive outlier models (SCNAO). The tabled entries are the MSEs of forecasts from the true value for each procedure

			Case 4				Case 5			
MA(1)	θ	n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	0.2	50	15.6876	15.5307	335.4637	291.3055	23.2724	22.3415	377.6177	305.3218
		100	6.5363	6.3402	331.3478	289.6748	10.8004	10.1524	376.7090	305.0210
		200	3.1783	3.0194	328.9628	288.4457	5.5499	5.1614	368.5186	292.2312
		500	1.2737	1.1464	324.0206	282.8793	2.1664	1.9714	363.5769	291.9760
	0.5	50	21.4561	19.9542	405.2880	354.9405	35.2868	32.4638	456.1099	368.2072
		100	10.1745	9.3605	403.5083	350.1487	17.0380	15.5046	452.0796	363.9333
		200	4.2445	3.8625	402.1661	349.8366	8.2719	7.4447	449.6205	361.7483
		500	1.8211	1.6390	399.1500	344.6099	3.2559	3.1257	449.3754	351.2203
	0.9	50	64.1014	61.5374	525.7548	403.8290	109.2872	106.0086	600.7119	414.0173
		100	35.5521	33.4190	506.1775	399.6097	55.6347	52.8529	576.6876	412.3471
		200	20.1401	18.7303	501.3439	396.5578	32.2498	29.0248	557.9922	410.8555
		500	8.5679	7.7968	464.3891	365.7485	18.0368	16.7742	524.6371	399.8471
			Case 4				Case 5			
MA(2)	(θ_1, θ_2)	n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.1,0.2)	50	22.5845	21.9069	452.9011	398.0433	35.2046	34.8525	509.0816	409.3684
		100	9.9744	9.4757	446.5758	388.5947	16.5321	16.0362	505.8922	409.2693
		200	4.6557	4.1901	444.7491	386.6467	7.9394	7.5424	505.8325	403.9200
		500	1.9017	1.7686	443.9539	371.9576	3.1742	2.8568	500.6349	393.5625
	(0.3,0.4)	50	27.5775	26.1986	408.1575	353.3388	40.3823	39.1708	456.0011	364.4920
		100	11.1174	10.0057	408.0187	351.9299	18.1523	17.2447	450.3721	363.3498
		200	4.6963	4.3676	400.2959	345.1057	8.3088	7.4779	447.9622	360.5343
		500	2.0377	2.0173	390.1950	341.0745	3.3329	3.0996	443.1273	346.9224
	(0.8,0.1)	50	28.4699	27.6158	340.3335	298.4588	40.4702	38.4467	378.3057	304.4032
		100	11.6891	11.1046	335.7040	295.2336	17.5859	15.8273	375.9615	301.1096
		200	5.1648	4.6483	331.4848	285.9192	7.7380	7.1963	372.9523	297.5728
		500	1.9534	1.8167	329.3151	279.3980	3.1277	3.0965	369.8844	289.1740
			Case 4				Case 5			
MA(3)	(θ_1, θ_2, θ_3)	n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.2)	50	19.1287	18.1723	340.5526	294.7477	31.1440	29.5868	379.4406	302.8680
		100	7.9159	7.1243	336.5798	292.2487	12.8990	11.6091	377.8335	301.7997
		200	3.5948	3.3432	336.0753	284.8166	5.9769	5.5586	374.9577	293.5094
		500	1.4423	1.4279	333.1032	280.4898	2.4137	2.3896	374.3454	286.7001
	(0.0,0.0,0.5)	50	38.7859	37.6223	410.4479	354.4821	62.4722	59.3486	461.5841	368.0566
		100	14.5627	13.8346	407.3904	346.9241	26.3127	23.6815	460.5585	361.6683
		200	7.0144	6.3129	400.5943	342.7700	10.8093	10.0526	458.5835	358.0211

Table 10: Continue

		500	2.5912	2.4098	395.0119	333.3757	4.3040	4.0888	455.6266	340.1282
	(0.0,0.0,0,0.9)	50	102.9580	101.9284	524.8869	393.7746	178.8003	173.4363	600.3338	406.0739
		100	65.8820	63.9055	516.9993	389.0108	125.8469	119.5545	575.5259	403.9608
		200	42.4839	40.3597	492.7151	383.8066	75.5600	68.0040	555.0891	390.5367
		500	12.3280	11.0952	459.4062	372.7658	20.3705	18.9445	514.8656	382.8626
			Case 4				Case 5			
MA(4)	(θ_1, θ_2, θ_3, θ_4)	n	MIS1	MIS2	IS	MLE	MIS1	MIS2	IS	MLE
	(0.0,0.0,0.0,0.2)	50	39.4643	36.7018	461.0675	362.7399	40.7378	37.8861	429.5653	316.3133
		100	5.4069	5.3529	338.0346	294.4542	9.9765	9.8768	397.6625	312.7948
		200	3.8317	3.6401	333.3379	287.4770	6.4538	6.1311	359.7480	279.5126
		500	1.4927	1.3434	288.5067	250.0024	2.0655	1.8590	343.7648	276.1500
	(0.0,0.0,0.0,0.5)	50	31.3517	29.7841	428.2714	362.2973	70.5432	69.8378	533.1112	362.5759
		100	19.3790	17.4411	413.6462	358.7626	29.4623	28.5785	450.5383	357.3415
		200	10.0623	9.3580	395.1882	348.3022	13.9938	13.2941	438.1190	354.8373
		500	2.7440	2.7165	309.8454	246.7455	4.3695	3.9326	408.6589	315.8453
	(0.0,0.0,0.0,0.9)	50	83.5388	82.7034	504.2850	413.8784	133.7948	127.1051	560.5889	399.5487
		100	61.3762	58.3074	503.4964	370.2890	79.7272	71.7545	556.4057	379.0456
		200	46.0344	41.4309	20480.9339	348.3849	92.1498	87.5424	503.3209	336.8654
		500	14.8598	14.7112	480.1576	331.6136	21.9924	19.7932	454.4644	315.4017

Table 11: Summary of IMA (1,1) model parameter estimates, estimated model variance, and one-step ahead forecasts MSD for the MIS1, MIS2, IS, and MLE analyses for the above-mentioned data

	MIS1	MIS2	IS	MLE
$\hat{\theta}$	0.08750062	0.09370597	0.09006741	0.08845628
Residuals variance $\hat{\sigma}^2$	38.66425000	38.60344000	52.62322000	52.36045000
Forecast	346.46300000	346.46200000	346.48800000	346.49600000
Forecasts MSD	0.21400410	0.21315640	0.23841050	0.24578280

Example

A real data set from (Box *et al.*, 2008) is used as an example to compare the forecasts made by the MIS1, MIS2, IS, and MLE procedures. This time series is the daily common stock closing prices of IBM (Series B) from May 17, 1961 - November 2, 1962. It consists of 369 observations. Box and Jenkins identified the series as an IMA(1,1) model, i.e., a first difference MA(1) model. We used the CRAN package tsoutliers to detect outliers in the series; see (de Lacalle, 2017) and (Chen and Liu, 1993). Three outliers were detected with IDs: 239, 258, and 270. The first outlier is a Temporary Change (TC) while the other two are Additive Outliers (AO).

We fit the first difference MA(1) model to this series using all four procedures: MIS1, MIS2, IS, and MLE. Table 11 displays the estimates of the MA(1) coefficient θ and the variances of the residuals for the procedures. We also computed one-step ahead forecasts of the last data point and the mean square difference between each forecast and that point.

The estimates of the MA(1) coefficient θ are similar. The MIS residual variances estimates are similar but both are much less than the residual variances of the IS and MLE fits. So the MIS fits are more precise than the IS and MLE fits. The forecasts of the four procedures are similar, also. The mean squared difference of the MIS procedures is slightly less than those of the IS and MLE procedures.

Conclusion

In this study, we have presented two robust modifications of the recursive Innovative Substitution (IS) algorithm for the fitting of the Moving Average (MA) time-series of order q. The IS algorithm is a series of Least Squares (LS) regressions. The first modification that we proposed, MIS1, replaces the last regression in the IS algorithm with a robust fit, while the second, MS2, replaces all the LS fits with robust fits. For the robust fit, we selected the High Breakdown (HBR) fit. This estimate has bounded influence in both the Y and X spaces, attains a 50% breakdown point, and is efficient for skewed as well as symmetrically distributed random errors.

The main goal of this study is to investigate the empirical properties of these modified IS procedures. Two large Monte Carlo studies were carried out to investigate respectively the efficiencies of the estimates and forecasts over a wide variety of MA models, MA(1) and MA(4), comparing them with the IS and MLE procedures. Random error distributions included the normal and symmetric as well as skewed contaminated normal distributions with various rates of contamination. These were formulated to produce both Innovative (IO) and Additive (AO) outliers.

Of course Not surprisingly, for normal situations, the MLE estimator was the most efficient. In most normal situations, though, the efficiencies of the MIS1 relative to the MLE exceeded 92%. For these normal situations, in general, the MIS1 was slightly more efficient than the

MIS2, while the MLE was slightly more efficient than the IS estimator. For the symmetric contaminated normal situations, the MIS2 estimator is generally 5 (500%) to 8 (800%) times more efficient than the MLE estimator. In skewed contaminated normal situations, the edge increases to 8-12 times more efficient than the MLE estimator. These are IO situations. For the AO situations, the MIS2 estimator is even more efficient. In many of these situations, it is 100 times more efficient than the MLE procedure. The MIS1 has high efficiency to the MLE estimator in all non-normal situations, also. The MIS2, however, is always more efficient than the MIS1 estimator. While the MIS2 estimator dominates the MIS1 estimator in terms of efficiency in non-normal situations, the edge is much less than in its comparisons with the MLE estimator. The results are essentially the same regardless of sample size.

For normal situations, the MLE forecasts are more efficient than those of the other procedures but its edge over the MIS forecasts is slight. In terms of forecasting, the MIS2 forecasts are slightly more efficient than those of the MIS1 procedure. For non-normal situations, the MIS forecasts are much more efficient than those of the MLE. For the AO situations, this edge increases, often to 100-fold. The MIS2 procedure dominates the MIS1 procedure in all situations but as in the normal case, the edge is slight.

The IS recursive algorithm for fitting a MA model is a series of LS regressions. In this study, we considered the simple modification of replacing these LS regressions with regressions based on a high breakdown, efficient estimator. We proposed two modifications, MIS1 (replaces only the last LS regression) and MIS2 (replaces all the LS regressions). In our simulations studies, both modifications showed high empirical efficiency for MA models with normally distributed random errors compared to the MLE estimator. For symmetric and skewed contaminated normal situations (IO and AO outliers), the MIS estimators were much more efficient than the MLE estimators. The results for our forecasting study were similar. For the non-normal situations, the MIS2 estimator was more efficient than the MIS1 estimator while for the normal error situations the MIS1 has a slight edge. In practice, of the two, we recommend the MIS2 estimator.

Author's Contributions

All authors equally contributed to this study.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and that no ethical issues are involved.

References

Box, G., Jenkins, G., & Reinsel, G. (2008), *Time series analysis: Forecasting and control*, *Hoboken, NJ*: John Wiley and Sons, Inc., 4th ed. doi.org/10.1002/9781118619193

Chang, W., McKean, J., Naranjo, J., & Sheather, S. (1999), "High-breakdown rank regression," *Journal of the American Statistical Association,* 205–219. doi.org/10.1080/01621459.1999.10473836

Chen, C., & Liu, L. (1993), "Joint estimation of model parameters and outlier effects in time series," *Journal of the American Statistical Association,* 88, 284-297. doi.org/10.1080/01621459.1993.10594321

de Lacalle., J. L. (2017), Tsoutliers: Detection of Outliers in Time Series, r package version 0.09.

Fuller, W. (1996), *Introduction to Statistical Time Series,* New York: John Wiley and Sons, 2nd ed. doi.org/10.1002/9780470316917

Hannan, E., & Rissanen, J. (1982), "Recursive estimation of mixed autoregressive-moving average order," Biometrika, 69, 81–94. doi.org/10.1093/biomet/69.1.81

Koreisha, S., & Pukkila, T. (1989), "Fast linear estimation methods for vector autoregressive moving average models," *Journal of Time Series Analysis,* 10, 325-339. doi.org/10.1111/j.1467-9892.1989.tb00032.x

Koul, H., & Saleh, A. (1993), "R-estimation of the parameters of autoregressive [AR(p)] models," *The Annals of Statistics*, 21, 534-551. doi.org/10.1214/aos/1176349041

Rousseeuw, P., & Van Driessen, K. (1999), "A fast algorithm for the minimum covariance determinant estimator," *Technometrics*, 41, 212–223. doi.org/10.1080/00401706.1999.10485670

Terpstra, J., McKean, J. W., & Naranjo, J. D. (2000), "Highly efficient weighted Wilcoxon estimates for autoregression," *Statistics*, 35, 45–80. doi.org/10.1080/02331880108802724