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Hopf Algebraic Structure of the (*p*,*q*)-Square Heizenberg White Noise Algebra

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Abstract: This study is devoted to solving the Hopf algebraic structure problem associated with the Fock realization of the (p, q)-square Heizenberg white noise algebra based on the two-parameter deformations of canonical commutation relations.

Keywords: Fock Realization, Hopf Algebraic Structure, (p, q)-Square Heizenberg White Noise Algebra

Introduction

The two-parameter quantum deformations algebras based on the Fock representation for the two parameters deformed quantum oscillator algebra obtained in (Riahi *et al*., 2020a) and its connection with the Meixner classes given in a series of papers (Berezansky, 1968; Berezansky and Kondratiev, 2013; Barhoumi and Riahi, 2010) which found a lot of interesting applications in quantum probability. The Hopf algebraic structure problem stated below has led, in the past 30 years, to a multiplicity of new results in different fields of mathematics and physics. The theory of multi-parameter quantum deformations of Lie algebras (Hu, 1999; Riahi *et al*., 2021), Lie bialgebras (Song and Su, 2006; Yue and Su, 2008), and quantization of Lie algebras (Chakrabarti and Jagannathan, 1991; Song *et al*., 2008; Su and Yuan, 2010) play an essential role in the quantum white noise literature.

More precisely, the Fock representation of two parameters deformed commutation relation was first studied by (Riahi *et al*., 2021) by constructing an interacting Fock space $F_{p,q}(\mathcal{H})$ as the space of representation.

(p,q)-Deformed Square Heizenberg White Noise Algebra

For $p,q \in$ R such that $0 < q < p \leq 1$, the (p,q)-Heizenberg algebra $\mathcal{H}_{p,q}$ is generated by Υ, Υ^+ and N satisfying the following relations:

$$\left[\Upsilon,\Upsilon^+\right]_q = \Upsilon\Upsilon^+ - q\Upsilon^+\Upsilon = p^N, \left[\Upsilon,\Upsilon^+\right]_p = q^N$$
$$\left[N,\Upsilon\right] = N\Upsilon - \Upsilon N = -\Upsilon, \left[N,\Upsilon^+\right] = N\Upsilon^+ - \Upsilon^+ N = \Upsilon^+, \Upsilon^* = \Upsilon^+$$

where for $\alpha \in \mathbb{R}$ the action of α^N on an element f_n with $n \in \mathbb{N}$ is given by:

$$\alpha^N f_n = \alpha^n f_n, \quad \alpha^N f_0 = f_0.$$

The first example is the one-mode realization of $\mathcal{H}_{p,q}$. If we put:

$$\Upsilon = D_{p,q}, \quad \Upsilon^+ = M_z, \quad N = z\frac{d}{dz},$$

where, M_z is the multiplication operator by $z \in \mathbb{C}$, in the space of all finite linear combinations of $f_n = z^n$, and $D_{p,q}$ is the (p,q)-derivative defined by:

$$\left(D_{p,q}f\right)(z) := \begin{cases} \dfrac{f(pz)-f(qz)}{(p-q)z}, & if\ z \neq 0 \\ f'(0), if\ z = 0. \end{cases}$$

Then the action of the generators on the basis vector f_n is obtained as follows:

$$\Upsilon f_n = [n]_{p,q} f_{n-1}, \quad \Upsilon^+ f_n = f_{n+1}, \quad N f_n = n f_n,$$

where, $[n]_{p,q}$ is the two parameters deformation of $n \in$ N, i.e.:

$$[n]_{p,q} = \sum_{i=1}^{n} q^{i-1} p^{n-i} = \frac{p^n - q^n}{p - q}.$$

As a consequence, we get:

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$$\left[D_{p,q}, M_z\right]_q f_n = p^n f_n, \left[D_{p,q}, M_z\right]_p f_n = q^n f_n,$$
$$\left[N, D_{p,q}\right] f_n = -D_{p,q} f_n, \left[N, M_z\right] f_n = M_z f_n,$$

and we conclude that the algebra generated by {1, $D_{p,q}$, M_z, N} gives a representation of $\mathcal{H}_{p,q}$. The second example is the infinite-dimensional analog representation. Define the operator $T_{p,q}$ on $\mathcal{H}^{\otimes n}$ by:

$$\mathcal{T}_{p,q}\left(\xi_1 \otimes \cdots \otimes \xi_n\right) = \sum_{\sigma \in S_n} q^{I(\sigma)} p^{C(\sigma)} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)},$$

where, $I(\sigma)$ and $C(\sigma)$ denotes respectively the number of inversions and conversions of the permutation $\sigma \in S_n$. Now we put:

$$\xi_1 \otimes_{p,q} \cdots \otimes_{p,q} \xi_n := \mathcal{T}_{p,q}\left(\xi_1 \otimes \cdots \otimes \xi_n\right), \ \xi_i \in \mathcal{H}, i \in \{1, \cdots, n\}.$$

Definition 2.1

Blitvi'c (2012) Define $\mathcal{F}_{p,q}^{(n)}(\mathcal{H}) := \mathcal{H}^{\otimes n}$ equipped with the following scalar product:

$$\left\langle \varphi^{(n)}, \psi^{(n)} \right\rangle_{p,q} := \left\langle \varphi^{(n)}, \psi^{(n)} \right\rangle_{\mathcal{F}_{p,q}^{(n)}(\mathcal{H})} = \left\langle \mathcal{T}_{p,q} \varphi^{(n)}, \psi^{(n)} \right\rangle. \quad (2.1)$$

The (p,q)-Fock space denoted by $F_{p,q}(\mathcal{H})$ is defined as:

$$\mathcal{F}_{p,q}(\mathcal{H}) = \bigotimes_{n=0}^{\infty} \mathcal{F}_{p,q}^{(n)}(\mathcal{H}).$$

For, $t \in \mathbb{R}$, let ∂_t and ∂_t^* be the pointwise annihilation and creation operators on $F_{p,q}(\mathcal{H})$ given by:

$$\partial_t f^{\otimes n} = [n]_{p,q} f(t) f^{\otimes(n-1)},$$
$$\partial_t^* f^{(n)} = \delta_t \hat{\otimes} f^{(n)},$$

where, δ_t is the delta function at t and stands for the symmetric tensor product. For each $\xi \in \mathcal{H}$, we define the creation operator $a^*(\xi)$ and the annihilation operator $a(\xi)$ on $F_{p,q}(\mathcal{H})$ as follows:

$$a(\xi) = \int_{\mathbb{R}} \xi(t) \partial_t dt, \ a^*(\xi) = \int_{\mathbb{R}} \xi(t) \partial_t^* dt, \ \forall \xi \in \mathcal{H}. \quad (2.2)$$

Proposition 2.2

Riahi *et al*. (2020b) The algebra generated by {1, $a^*(\xi)$, $a(\xi)$, N} give a fock realization of the (p, q)-Heizenberg algebra, i.e.:

$$\left[a(\xi), a^*(\eta)\right]_q = p^N \langle \xi, \eta \rangle .1, \ \xi, \eta \in \mathcal{H} \quad (2.3)$$

$$\left[N, a^*(\xi)\right] = a^*(\xi), \ \left[N, a(\xi)\right] = -a(\xi) \quad (2.4)$$

where, N is the preservation operator, i.e.:

$$N f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n = n f_1 \otimes_{p,q} \cdots \otimes_{p,q} f_n.$$

Definition 2.3:

The algebra with generators $a(\xi)$, $a^*(\xi)$, $N, \xi \in \mathcal{H}$, and a central element I with commutation relations (2.3)-(2.4) is called the (p, q) Heizenberg white noise algebra.

Let ∂_{et} be the operator defined by:

$$\tilde{\partial}_t f^{\otimes n} = n f(t) f^{\otimes(n-1)}.$$

and put:

$$B_t := \partial_t^2, \ B_t^* := \left(\partial_t^*\right)^2, \ N_t := \partial_t^* \tilde{\partial}_t.$$

For $\xi \in \mathcal{H}$, we define:

$$B_{\rightarrow} = \int_{\mathbb{R}} \xi(t) B_t dt, \quad B_\xi^* = \int_{\mathbb{R}} \xi(t) B_t^* dt, \quad N_\xi = \int \xi(t) N_t dt. \quad (2.5)$$

It is known from (Accardi *et al*., 1999) that:

$$\delta^2(t) = c\delta(t), \quad (2.6)$$

where, $c \in \mathbb{C}$ is an arbitrary constant.

Theorem 2.4

Riahi *et al*. (2021) The operators B_ξ, B_η^* and N_ξ satisfy the following commutations relations:

$$\left[B_\xi, B_\eta^*\right]_q = 2c\langle \xi, \eta \rangle .1 + \left(2 + q + p^{N_{\xi\eta}}\right)\left[N_{\xi\eta}\right]_{p,q}, \quad (2.7)$$

$$\left[B_\xi, B_\eta^*\right]_q = 2c\langle \xi, \eta \rangle .1 + \left(2 + p + q^{N_{\xi\eta}}\right)\left[N_{\xi\eta}\right]_{p,q}, \quad (2.8)$$

$$\left[N_\xi, B_\eta^*\right] = 2B_{\xi\eta}^*, \ \left[N_\xi, B_\eta\right] = -2B_{\xi\eta}, \quad (2.9)$$

$$\left[N_\xi, N_\eta\right] = \left[B_\xi, B_\eta\right] = \left[B_\xi^*, B_\eta^*\right] = 0, \ \xi, \eta \in \mathcal{H}. \quad (2.10)$$

Remark 2.5

One can see that if $p = 1$ and $q \to 1$, the commutation relations (2.7) - (2.10) become:

$$\begin{aligned} &\left[B_\xi, B_\eta^*\right] = 2c\langle \xi, \eta \rangle .1 + 4N_{\xi\eta}, \\ &\left[B_\xi, B_\eta^*\right] = 2c\langle \xi, \eta \rangle .1 + 4N_{\xi\eta}, \\ &\left[N_\xi, B_\eta^*\right] = 2B_{\xi\eta}^*, \quad \left[N_\xi, B_\eta\right] = -2B_{\xi\eta}, \\ &\left[N_\xi, N_\eta\right] = \left[B_\xi, B_\eta\right] = \left[B_\xi^*, B_\eta^*\right] = 0. \end{aligned}$$

This shows that the (*p*,*q*)-square white noise algebra gives the square white noise Lie algebra when $p = 1$ and $q \to 1$.

Definition 2.6

The algebra with generators $B_\xi, B_\eta^*, N_\xi, \xi \in \mathcal{H}$ and a central element 1 with commutation relations (2.7)–(2.10) is called the (*p*, *q*)-square Heizenberg white noise algebra denoted by $Sl_{2,p,q}(\mathcal{H})$.

Hopf Structure of the (*p*,*q*)-Square Heizenberg White Noise Algebra

Lemma 3.1

Let Ψ_1, $\Psi 2$, Ψ_3, and Ψ_4 be four non-zero continuous functions satisfying the following conditions:

$$\begin{cases} \Psi_1\Psi_3 = 1, \ \Psi_2\Psi_4 = 1 \\ \Psi_4(n+1)\Psi_1(m) = q\Psi_4(n)\Psi_1(m-2) \ \ \forall n,m \in \mathbb{N}, \\ \Psi_2(n)\Psi_3(m+1) = q\Psi_2(n-2)\Psi_3(m) \ \ \forall n,m \in \mathbb{N}, \end{cases} \qquad (3.1)$$

and for any non-zero continuous function f, define $\Delta(f(N))$ by:

$$\Delta(f(N)) = f(N)\hat{\otimes}1 + 1\hat{\otimes} f(N). \qquad (3.2)$$

Then there exists a unique homomorphism:

$$\Delta : Sl_{2,p,q}(\mathcal{H}) \to Sl_{2,p,q}(\mathcal{H}) \times Sl_{2,p,q}(\mathcal{H})$$

with:

$$\Delta(N) = N\hat{\otimes}1 + 1\hat{\otimes} N, \qquad (3.3)$$

$$\Delta(B_\eta^*) = B_\eta^*\hat{\otimes}\Psi_1(N) + \Psi_2(N)\hat{\otimes} B_\eta^*, \qquad (3.4)$$

$$\Delta(B_\xi) = B_\xi\hat{\otimes}\Psi(N) + \Psi_4(N)\hat{\otimes} B_\xi. \qquad (3.5)$$

Proof

By direct calculations we have:

$$\begin{aligned} &\Delta(B_\xi)\Delta(B_\eta^*) = \left(B_\xi\hat{\otimes}\Psi_3(N) + \Psi_4(N)\hat{\otimes} B_\xi\right) \\ &\left(B_\eta^*\hat{\otimes}\Psi_1(N) + \Psi_2(N)\hat{\otimes} B_\eta^*\right) \\ &= B_\xi B_\eta^*\hat{\otimes}\Psi_3(N)\Psi_1(N) + B_\xi\Psi_2(N)\hat{\otimes}\Psi_3(N)B_\eta^* \\ &+ \Psi_4(N)B_\eta^*\hat{\otimes}B_\xi\Psi(N) + \Psi_4(N)\xi_2(N)\hat{\otimes}B_\xi B_\eta^*. \end{aligned}$$

and:

$$\begin{aligned} &\Delta(B_\eta^*)\Delta(B_\xi) = \left(B_\eta^*\hat{\otimes}\Psi_1(N) + \Psi_2(N)\hat{\otimes} B_\eta^*\right) \\ &\left(B_\xi\hat{\otimes}\Psi_3(N) + \Psi_4(N)\hat{\otimes} B_\xi\right) \\ &= B_\eta^* B_\xi\hat{\otimes}\Psi_1(N)\Psi_3(N) + B_\eta^*\Psi_4(N)\hat{\otimes}\Psi_1(N)B\xi \\ &+ \Psi_2(N)B_\xi\hat{\otimes}B_\eta^*\Psi_3(N) + \Psi_2(N)\Psi_4(N)\hat{\otimes}B_\eta^* B_\xi. \end{aligned}$$

Then by using (2.7) we obtain:

$$\begin{aligned} &\Delta(B_\xi)\Delta(B_\eta^*) - q\Delta(B_\eta^*)\Delta(B_\xi) \\ &= \left(2c\langle\xi,\eta\rangle.1 + \left(2+q+p^{N\xi\eta}\right)\left[N_{\xi\eta}\right]_{P,q}\right)\hat{\otimes}\Psi_1(N)\Psi_3(N) \\ &+ \left(\Psi_4(N)B_\eta^*\hat{\otimes}B_\xi\Psi_1(N) - qB_\eta^*\Psi_4(N)\hat{\otimes}\Psi_1(N)B_\xi\right) \\ &+ \left(B_\xi\Psi_2(N)\hat{\otimes}\Psi_3(N)B_\mu^* - q\Psi_2(N)B_\xi\hat{\otimes}B_\eta^*\Psi_3(N)\right) \\ &+ \Psi_2(N)\Psi_4(N)\hat{\otimes}\left(2c\langle\xi,\eta\rangle.1 + \left(2+q+p^{N\xi\eta}\right)\left[N_{\xi\eta}\right]_{p,q}\right). \end{aligned}$$

Moreover, using a basis $(\zeta_k)_k$ of $\mathcal{H}$, (2.5) and (2.6) give:

$$\begin{aligned} B_\eta^*\varsigma_k^{\otimes n} &= c\eta\otimes\varsigma_k^{\otimes n} \\ B_\xi\varsigma_k^{\otimes n} &= [n]_{p,q}[n-1]_{p,q}\left\langle\xi,\varsigma_k^2\right\rangle\varsigma_k^{\otimes(n-2)}. \end{aligned}$$

Hence we obtain:

$$\begin{aligned} &\left(\Psi_4(N)B_\eta^*\hat{\otimes}B_\xi\Psi_1(N) - qB_\eta^*\Psi_4(N)\hat{\otimes}\Psi_1(N)B_\xi\right)\varsigma_k^{\otimes n}\hat{\otimes}\varsigma_k^{\otimes m} \\ &= \Psi_4(N)\left(c\eta\hat{\otimes}\varsigma_k^{\otimes n}\right)\otimes\Psi_1(m)B_\xi\varsigma_k^{\otimes m} \\ &- q\Psi_4(n)B_\eta^*\varsigma_k^{\otimes n}\hat{\otimes}\Psi_1(N)\left([m]_{p,q}[m-1]_{p,q}\left\langle\xi,\varsigma_k^2\right\rangle\varsigma_k^{\otimes(m-2)}\right) \\ &= \left(\Psi_4(n+1)\Psi_1(m) - q\Psi_4(n)\Psi_1(m-2)\right) \\ &\times c[m]_{p,q}[m-1]_{p,q}\left\langle\xi,\varsigma_k^2\right\rangle\eta\hat{\otimes}\varsigma_k^{\otimes}\hat{\otimes}\varsigma_k^{\otimes(m-2)} \\ &= 0. \end{aligned}$$

and:

$$\begin{aligned} &\left(B_\xi\Psi_2(N)\otimes\Psi_3(N)B_\eta^* - q\Psi_2(N)B_\xi\otimes B_\eta^*\Psi_3(N)\right)\zeta_k^{\otimes n}\hat{\otimes}\zeta_k^{\otimes m} \\ &= \Psi_2(n)B_\xi\zeta_k^{\otimes n}\hat{\otimes}\Psi_3(N)\left(c\eta\otimes\zeta_k^{\otimes m}\right) \\ &-q\Psi_2(N)\left([n]_{p,q}[m-1]_{p,q}\left\langle\xi,\zeta_k^2\right\rangle\zeta_k^{\otimes(n-2)}\right)\hat{\otimes}\Psi_3(m)B_\eta^*\zeta_k^{\otimes m} \\ &= \left(\Psi_3(m+1)\Psi_2(n) - q\Psi_3(m)\Psi_2(n-2)\right) \\ &\times c[n]_{p,q}[n-1]_{p,q}\left\langle\xi,\zeta_k^2\right\rangle\zeta_k^{\otimes(n-2)}\hat{\otimes}\eta\hat{\otimes}\zeta_k^{\otimes m} \\ &= 0 \end{aligned}$$

Hence, using (3.1) we get:

$$\begin{aligned} &\Delta(B_\xi)\Delta(B_\eta^*) - q\Delta(B_\eta^*)\Delta(B_\xi) \\ &= \left(2c\langle\xi,\eta\rangle.1 + \left(2+q+p^{N\xi\eta}\right)\left[N_{\xi\eta}\right]_{p,q}\right)\hat{\otimes}1 \\ &+1\otimes\left(2c\langle\xi,\eta\rangle.1 + \left(2+q+p^{N\xi\eta}\right)\left[N_{\xi\eta}\right]_{p,q}\right) \\ &= \Delta\left(2c\langle\xi,\eta\rangle.1 + \left(2+q+p^{N\xi\eta}\right)\left[N_{\xi\eta}\right]_{p,q}\right), \end{aligned}$$

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and by similar calculations we obtain:

$$\left[\Delta(N),\Delta\left(B_\eta^*\right)\right]=\Delta\left(2B_\eta^*\right),\quad \left[\Delta(N),\Delta\left(B_\eta\right)\right]=-\Delta\left(2B_\eta\right).$$

Which gives the statement.

Lemma 3.2

If $\Psi_1=\Psi_2$ and $\Psi_3=\Psi_4$, then Δ is associative, i.e.:

$$\left(1\hat{\otimes}\Delta\right)\Delta=\left(\Delta\hat{\otimes}1\right)\Delta.$$

Proof

By using (3.4), we get:

$$\begin{aligned}\left(1\hat{\otimes}\Delta\right)\Delta\left(B_\eta^*\right)&=\left(1\hat{\otimes}\Delta\right)\left(B_\eta^*\hat{\otimes}\Psi_1(N)+\Psi_2(N)\hat{\otimes}B_\eta^*\right)\\&=B_\eta^*\hat{\otimes}\Delta\left(\Psi_1(N)\right)+\Psi_2(N)\hat{\otimes}\Delta\left(B_\eta^*\right)\\&=B_\eta^*\hat{\otimes}\left(\Psi_1(N)\hat{\otimes}1+1\hat{\otimes}\Psi_1(N)\right)\\&\quad+\Psi_2(N)\hat{\otimes}\left(B_\eta^*\hat{\otimes}\Psi_1(N)+\Psi_2(N)\hat{\otimes}B_\eta^*\right)\\&=B_\eta^*\hat{\otimes}\Psi_1(N)\hat{\otimes}1+B_\eta^*\hat{\otimes}1\hat{\otimes}\Psi_1(N)\\&\quad+\Psi_2(N)\hat{\otimes}B_\eta^*\hat{\otimes}\Psi_1(N)+\Psi_2(N)\hat{\otimes}\Psi_2(N)\hat{\otimes}B_\eta^*\end{aligned}$$

and:

$$\begin{aligned}\left(\Delta\hat{\otimes}1\right)\Delta\left(B_\eta^*\right)&=\left(\Delta\hat{\otimes}1\right)\left(B_\eta^*\hat{\otimes}\Psi_1(N)+\Psi_2(N)\hat{\otimes}B_\eta^*\right)\\&=\Delta\left(B_\eta^*\right)\hat{\otimes}\Psi_1(N)+\Delta\left(\Psi_2(N)\right)\hat{\otimes}B_\eta^*\\&=\left(B_\eta^*\hat{\otimes}\Psi_1(N)+\Psi_2(N)\otimes B_\eta^*\right)\hat{\otimes}\Psi_1(N)\\&\quad+\left(\Psi_2(N)\hat{\otimes}1+1\hat{\otimes}\Psi_2(N)\right)\otimes B_\eta^*\\&\quad+\Psi_2(N)\hat{\otimes}1\hat{\otimes}B_\eta^*+1\hat{\otimes}\Psi_2(N)\hat{\otimes}B_\eta^*.\end{aligned}$$

Thus, using a basis $(\zeta_k)_k$ of $\mathcal{H}$, we obtain:

$$\begin{aligned}\left(1\hat{\otimes}\Delta\right)\Delta\left(B_\eta^*\right)\zeta_k^{\otimes n}\hat{\otimes}\zeta_k^{\otimes m}\hat{\otimes}\zeta_k^{\otimes s}&=c\eta\hat{\otimes}\zeta_k^{\otimes n}\hat{\otimes}\Psi_1(m)\zeta_k^{\otimes m}\hat{\otimes}\zeta_k^{\otimes s}\\&\quad+c\eta\hat{\otimes}\zeta_k^{\otimes n}\hat{\otimes}\zeta_k^{\otimes m}\hat{\otimes}\Psi_1(s)\zeta_k^{\otimes s}\\&\quad+\Psi_2(n)\zeta_k^{\otimes n}\hat{\otimes}c\eta\otimes\zeta_k^{\otimes m}\hat{\otimes}\Psi_1(s)\zeta_k^{\otimes s}\\&\quad+\Psi_2(n)\zeta_k^{\otimes n}\hat{\otimes}\Psi_2(m)\zeta_k^{\otimes n}\hat{\otimes}c\eta\otimes\zeta_k^{\otimes s}\end{aligned}$$

and:

$$\begin{aligned}\left(\Delta\hat{\otimes}1\right)\Delta\left(B_\eta^*\right)\zeta_k^{\otimes n}\hat{\otimes}\zeta_k^{\otimes m}\hat{\otimes}\zeta_k^{\otimes s}&=\Psi_1(n)\zeta_k^{\otimes n}\hat{\otimes}c\eta\otimes\zeta_k^{\otimes m}\hat{\otimes}\Psi_1(s)\zeta_k^{\otimes s}\\&\quad+\Psi_1(n)\zeta_k^{\otimes n}\hat{\otimes}\Psi_2(m)\zeta_k^{\otimes n}\hat{\otimes}c\eta\otimes\zeta_k^{\otimes s}\\&\quad+c\eta\hat{\otimes}\zeta_k^{\otimes n}\hat{\otimes}\Psi_2(m)\zeta_k^{\otimes m}\hat{\otimes}\zeta_k^{\otimes s}\\&\quad+c\eta\hat{\otimes}\zeta_k^{\otimes n}\hat{\otimes}\zeta_k^{\otimes m}\hat{\otimes}\Psi_2(s)\zeta_k^{\otimes s}.\end{aligned}$$

Hence, if $\Psi_1=\Psi_2$, we conclude that:

$$\left(1\hat{\otimes}\Delta\right)\Delta\left(B_\eta^*\right)=\left(\Delta\hat{\otimes}1\right)\Delta\left(B_\eta^*\right).$$

Similarly, if $\Psi_3=\Psi_4$, one can see that:

$$\left(1\hat{\otimes}\Delta\right)\Delta\left(B_\xi\right)=\left(\Delta\hat{\otimes}1\right)\Delta\left(B_\xi\right).$$

and:

$$\left(1\hat{\otimes}\Delta\right)\Delta(N)=\left(\Delta\hat{\otimes}1\right)\Delta(N).$$

Which completes the proof.

Lemma 3.3

There exist a unique homomorphism:

$$\varepsilon: Sl_{2,p,q}(\mathcal{H})\rightarrow\mathbb{C}$$

with:

$$\varepsilon(1)=1\ and\ \ \varepsilon(N)=\varepsilon\left(B_\xi\right)=\varepsilon\left(B_\xi^*\right)=0,$$

such that the following diagrams are commutative:

$$\begin{array}{ccc}Sl_{2,p,q}(\mathcal{H}) & \xrightarrow{\Delta} & Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\\ \downarrow id & & \downarrow 1\otimes\varepsilon\\ Sl_{2,p,q}(\mathcal{H}) & \xrightarrow{\pi_1} & Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\end{array}\qquad(3.6)$$

$$\begin{array}{ccc}Sl_{2,p,q}(\mathcal{H}) & \xrightarrow{\Delta} & Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\\ \downarrow id & & \downarrow \varepsilon\otimes 1\\ Sl_{2,p,q}(\mathcal{H}) & \xrightarrow{\pi_2} & Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\end{array}\qquad(3.7)$$

where, π_1 (resp. π_2) denotes the isomorphism u 7 → u ⊗ 1 (resp. u 7→ 1 ⊗ u).

Proof

It is clair that $(\varepsilon(1),\varepsilon(N),\varepsilon(B_\xi),\varepsilon(B_\eta{}^*))=(1,0,0,0)$ satisfies (2.7)– (2.10). So we have the algebraic homomorphism ε. Moreover, the commutativity of (3.6) and (3.7) can be easily verified.

Lemma 3.4

If $\Psi_1\Psi_3=1$ and $\Psi_2\Psi_4=1$, then there exists a unique linear map S of $Sl_{2,p,q}(\mathcal{H})$ such that:

$$\begin{cases}S\left(B_\eta^*\right)=-\Psi_2(N)B_\eta^*\Psi_3(N),\\ S\left(B_\xi\right)=-\Psi_1(N)B_\xi\Psi_4(N),\\ S(N)=-N,S(I)=I,S\left(f(N)\right)=f(N),\end{cases}\qquad(3.8)$$

where, f is a continuous non-zero arbitrary function.

Proof

Using (2.3), we get:

$$\begin{aligned}&S(B_\xi)S(B_\eta^*)-qS(B_\eta^*)S(B_\xi)\\&=(-\Psi_1(N)B_\xi\Psi_4(N))(-\Psi_2(N)B_\eta^*\Psi_3(N))\\&-q(-\Psi_2(N)B_\eta^*\Psi_3(N))(-\Psi_1(N)B_\xi\Psi_4(N))\\&=\Psi_1(N)B\xi B\eta*\Psi_3(N)-q\Psi_2(N)B\eta*B_\xi\Psi_4(N).\end{aligned}$$

Moreover, it is easy to see that:

$$\Psi_1(N)B_\xi B_\eta^*\Psi_3(N)-q\Psi_2(N)B_\eta^* B_\xi\Psi_4(N),$$

and $2c\langle\xi,\eta\rangle.1+(2+q+p^{N\xi\eta})[N_{\xi\eta}]_{p,q}$ coincide on the basis $(\zeta_k)_k$ of $\mathcal{H}$. Hence we conclude that:

$$\begin{aligned}&S(B_\xi)S(B_\eta^*)-qS(B_\eta^*)S(B_\xi)\\&=S\left(\left(2c\langle\xi,\eta\rangle.1+\left(2+q+p^{N\xi\eta}\right)\left[N_{\xi\eta}\right]p,q\right)\right),\end{aligned}$$

this proves that S preserves (2.7). similarly, we can easily verify that (2.8) (2.10) are also preserved by S. Thus there exist a homomorphism satisfying (3.8):

$$S: S1_{2,p,q}(\mathcal{H})\to S1_{2,p,q}(\mathcal{H})$$

Lemma 3.5

The following diagrams are commutative:

$$\begin{aligned}&Sl_{2,p,q}(\mathcal{H})\xrightarrow{\Delta}Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\\&\downarrow\sigma\text{o}\varepsilon\qquad\downarrow 1\otimes S\\&\mathcal{H}_{p,q}\xrightarrow{m^{-1}}Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\\&Sl_{2,p,q}(\mathcal{H})\xrightarrow{\Delta}Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\\&\downarrow\sigma\text{o}\varepsilon\qquad\downarrow 1\otimes S\\&Sl_{2,p,q}(\mathcal{H})\xrightarrow{m^{-1}}Sl_{2,p,q}^{\otimes 2}(\mathcal{H})\end{aligned}\qquad(3.9)$$

where:

$$m(u\otimes\upsilon)=u\upsilon\ \forall u,\upsilon\in Sl_{2,p,q}(\mathcal{H}),$$

and σ: ℂ ⟶ $Sl_{2,p,q}(\mathcal{H})$ such that:

$$\sigma(a)=a1.$$

Proof

Using (3.3), (3.4), and (3.5) we get:

$$\begin{aligned}&m\text{o}(S\,\hat{\otimes}1)\text{o}\Delta(N)=m\text{o}(S\,\hat{\otimes}1)(N\,\hat{\otimes}1+1\hat{\otimes}N)\\&=m\text{o}(S(N)\hat{\otimes}1+S(1)\hat{\otimes}N)\\&=m\text{o}(-N\,\hat{\otimes}1+1\hat{\otimes}N)=0=\sigma\text{o}\varepsilon(N),\end{aligned}$$

$$\begin{aligned}&m\text{o}(S\,\hat{\otimes}1)\text{o}\Delta(B_\eta^*)=m\text{o}(S\,\hat{\otimes}1)(B_\eta^*\,\hat{\otimes}\,\Psi_1(N)+\Psi_2(N)\hat{\otimes}B_\eta^*)\\&=m\text{o}(S(B_\eta^*)\hat{\otimes}\Psi_1(N)+S(\Psi_2(N))\hat{\otimes}B_\eta^*)\\&=m\text{o}(-\Psi_2(N)A^+\Psi_3(N)\hat{\otimes}\Psi_1(N)+\Psi_2(N)\hat{\otimes}B_\eta^*)\\&-\Psi_2(N)B_\eta^*+\Psi_2(N)B_\eta^*\\&=0=\sigma\text{o}\varepsilon(B_\eta^*),\end{aligned}$$

$$\begin{aligned}&m\text{o}(S\,\hat{\otimes}1)\text{o}\Delta(B_\xi)=m\text{o}(S\,\hat{\otimes}1)(B_\xi\,\hat{\otimes}\,\Psi_3(N)+\Psi_4(N)\hat{\otimes}B_\xi)\\&=m\text{o}(S(B_\xi)\hat{\otimes}\Psi_3(N)+S(\Psi_4(N))\hat{\otimes}B_\xi)\\&=m\text{o}(-\Psi_4(N)B_\xi\Psi_1(N)\hat{\otimes}\Psi_3(N)+\Psi_4(N)\hat{\otimes}B_\xi)\\&-\Psi_4(N)B_\xi+\Psi_4(N)B_\xi\\&=0=\sigma\text{o}\varepsilon(B_\xi).\end{aligned}$$

Similarly, we have:

$$m\text{o}(1\hat{\otimes}S)\text{o}\Delta(\Xi)=\sigma\text{o}\varepsilon(X),\quad \Xi=N,B_\xi^*,B_\xi,I,$$

and (3.9) is established.

Theorem 3.6

Suppose that Ψ_1, Ψ_2, Ψ_3, and Ψ_4 satisfy the following conditions:

$$\begin{cases}\Psi_1\Psi_3=1,\Psi_2\Psi_4=1,\\\Psi_1=\Psi_2,\Psi_3=\Psi_4,\\\Psi_4(n+1)\Psi_1(m)=q\Psi_1(n)\Psi_1(m-2)\ \ \forall n,m\in\mathbb{N}.\end{cases}$$

Then, $\left(Sl_{2,p,q}(\mathcal{H}),\Delta,\varepsilon,S\right)$ is a Hopf algebra.

Proof

So by collecting Lemmas 3.1-3.5 we obtain:

$$\begin{aligned}&(\Delta\,\hat{\otimes}1)\text{o}\Delta=(1\hat{\otimes}\Delta)\text{o}\Delta(\textit{coassociativity})\\&(\varepsilon\,\hat{\otimes}1)\text{o}\Delta=(1\hat{\otimes}\varepsilon)\text{o}\Delta=id\,(\textit{counitary})\\&m\text{o}(S\,\hat{\otimes}1)\text{o}\Delta=m\text{o}(1\hat{\otimes}S)\text{o}\Delta=\sigma\text{o}\varepsilon.\end{aligned}$$

Hence we conclude that $\left(Sl_{2,p,q}(\mathcal{H}),\Delta,\varepsilon,S\right)$ is a Hop f algebra.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and that no ethical issues are involved.

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