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Positive Solutions for Some Weighted Elliptic Problems

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Abstract: In this study, we study the existence and the nonexistence of positive solutions for the following nonlinear elliptic problems:

$$-div\left(a(x)\nabla u\right)=f(x,u),\ u\in H_0^1(\Omega) \qquad \text{(P)}$$

where, Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geq 2$, $a(x)$ is a smooth function on $\bar{\Omega}$ and $f(x, s)$ is asymptotically linear in s at infinity, that is $\lim_{s\to+\infty}\frac{f(x,s)}{s}=\ell<\infty$. We will prove that the problem (P) has a positive solution for ℓ large enough and does not have positive solutions for ℓ less than the first eigenvalue of the operator. We prove also that the method works for the case when $f(x, s)$ is sub-critical and super-linear at $+\infty$.
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Introduction

The advantage of nonlinear equations lies in its stability to explain the evolution of a problem. In this study, we consider the following nonlinear elliptic weighted problems:

$$\begin{cases}-div\left(a(x)\nabla u\right)=f(x,u), & in\ \ \Omega,\\ \qquad u=0 & on\ \ \partial\Omega,\end{cases} \tag{1.1}$$

where, Ω is a regular bounded domain in $\mathbb{R}^N$, $N \geq 2$, $a(x)$ is a continuous function on and $f(x, s)$ is asymptotically linear in s at infinity, that is:

$$\lim_{s\to+\infty}\frac{f(x,s)}{s}=\ell<\infty.$$

This type of equations was proposed by (Turing, 1952) for modeling morphogenesis phenomena in Biology, in population dynamics. It was a model of interaction of species or chemicals given by:

$$\frac{\partial u}{\partial t}=div\left(a(x)\nabla u\right)=f(x,u),$$

where, u is the density, $\mathrm{div}(a(x)\nabla u)$ represents the substance of diffusion through the system and f models the interaction of substances. In the stationary case and when $f(x, s)$ depends only on s and f is asymptotically linear at $+\infty$, the problem (1.1) was studied by (Saanouni and Trabelsi, 2016a). They considered the problem:

$$\begin{cases}-div\left(a(x)\nabla u\right)=\lambda f(u), & in\ \ \Omega,\\ \qquad u=0 & on\ \ \partial\Omega,\end{cases} \tag{1.2}$$

as a generalisation of the work done by (Mironescu and Rădulescu, 1993; 1996; Rădulescu, 2008; Sanchón, 2007) where a is constant. They supposed that f: $\mathbb{R}\to\mathbb{R}$ is a positive, increasing and convex smooth function and the condition $f(0) > 0$ is capital in their study.

When f is super-linear ($l = +\infty$) and a is constant, the problem was studied in (Brezis *et al*., 1996; Martel, 1997). With the same nonlinearity and a constant, the problem was generated to the p-Laplace operator in (Filippakis and Papageorgiou, 2006; Schechter, 1995). Also, the same problems have been studied with the Bi-Laplacian see for example (Arioli *et al*., 2005; Abid *et al*., 2008; Saanouni and Trabelsi, 2016b; Wei, 1996).

In a recent work, (Li and Huang, 2019) a generalized quasilinear Schrdinger equations with asymptotically linear nonlinearities. They supposed that the nonlinearities $h(s)$ depend only on s and they prove the existence of positive solutions via variational methods. For the superlinear nonlinearities, the quasilinear

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Schrdinger equations was investigated by (Li *et al*., 2020). Zhou (2002) the problem (1.1) was treated when *a* is constant by Zhou but with more general nonlinearities: $f(x, s)$ depends on x and s and $f(x, 0) = 0$. More precisely, He supposed that the reaction function $f(x, t)$ satisfies the following conditions:

- (F1) $f(x, t) \in C(\bar{\Omega}\times\mathbb{R})$, $f(x, t) \geq 0$ for all $t > 0$ and $x \in \bar{\Omega}$ and $f(x, t) \equiv 0$ for $t \leq 0$ and $x \in \bar{\Omega}$
- (F2) $\lim_{t\to 0}\frac{f(x,t)}{t} = p(x)$, $\lim_{t\to+\infty}\frac{f(x,t)}{t} = \ell < \infty$ uniformly in a.e.$x \in\Omega$, where $0 \leq p(x) \in L^{\infty}(\Omega)$ and $\|p(x)\|_{\infty} < \lambda_1$, where $\lambda_1 > 0$ is the first eigenvalue of $(-\mathrm{div}(a(x)\nabla.),\ H_0^1(\Omega))$.
- (F3) The function $\frac{f(x,t)}{t}$ is nondecreasing with respect to $t > 0$; for *a.e.*$x\in\Omega$.

In this study, we will consider the problem (1.1) when the function a is not constant and use the scheme as in (Zhou, 2002) to study the existence of positive solution for the problem (1.1) and extend its results.

Definition 1.1

We say that u is a solution of (1.1) if $u \in H_0^1(\Omega)$ and:

$$\int_{\Omega} a(x)\nabla u\nabla\varphi = \int_{\Omega} f(x,u)\varphi,\ \ for\,all\,\varphi\in H_0^1(\Omega). \tag{1.3}$$

In order to find a solution to problem (1:1), we can look up for a critical point of the C^1 functional I defined on $H^1(\Omega)$ by:

$$I(u) = \frac{1}{2}\int_{\Omega} a(x)|\nabla u|^2\,dx - \int_{\Omega} F(x,u)dx \tag{1.4}$$

where:

$$F(x,s) = \int_0^s f(x,t)dt.$$

By the condition ($F1$) and the strong maximum principle, a nonzero critical point of the functional I is a positive solution of problem (1.1). Our approach to prove the existence of nonzero critical point of I is using the Mountain Pass Theorem given by (Ambrosetti and Rabinowitz, 1973; Rabinowitz, 1986). Often, one requires the technical condition introduced in (Ambrosetti and Rabinowitz, 1973), that is: for some $\theta > 2$ and $M > 0$:

$$0 < \theta F(x,t) \leq f(x,t)t,\ \ for\,all\,|t| \geq M\ \ and\ \ x\in\Omega. \tag{AR}$$

Condition (AR) was an important way to prove that a functional satisfies the compactness condition, that is each Palais Smale sequence is bounded and then relatively compact. But the condition (AR) implies that $\lim_{t\to+\infty}\frac{f(x,t)}{t^2} = +\infty$ and so the function $f(x, t)$ is super-linear with respect to t at infinity: $\lim_{t\to+\infty}\frac{f(x,t)}{t} = +\infty$ which contradicts the fact that f is asymptotically linear at $+\infty$. So, in this study we can not suppose the condition (AR).

In some papers and in order to avoid condition (AR), some other conditions were made as in (Costa and Magalhaes, 1994; Costa and Miyagaki, 1995; Jeanjean, 1999; Schechter, 1995; Stuart and Zhou, 1996; Stuart and Zhou, 1999; Zhou, 2002) and the references therein. In this study we will applyMountain Pass Theorem without any equivalent assumption.

Main Results

Before setting our main results, we introduce some notations and assumptions. First, we suppose that the continuous function $a(x)$ satisfies the condition:

$$0 < a_1 \leq a(x) \leq a_2,\ for\ some\ positive\ constants\ a_1\ and\ a_2\ and\ a.e\ x\in\Omega. \tag{A}$$

We denote by $\|.\|_p$ the $L^p(\Omega)$-norm and by $\|.\|$ the norm of $H_0^1(\Omega)$ induced by the inner product $< u, v >= \int_{\Omega} a(x)\,\nabla u.\nabla v\,dx$ for $u, v \in H_0^1(\Omega)$, that is:

$$\|u\| = \left(\int_{\Omega} a(x)|\nabla u|^2\,dx\right)^{\frac{1}{2}}.$$

We denote φ_1 a normalised positive eigenfunction associated to λ_1, that is:

$$\begin{cases} -div\left(a(x)\nabla\varphi_1\right) = \lambda_1\varphi_1 \ \ in\ \Omega, \\ \varphi_1 = 0 \quad on\ \partial\Omega, \\ \|\varphi_1\|_2 = 1. \end{cases}$$

We will prove the following result when the nonlinear term is asymptotically linear.

Theorem 2.1

Let Ω be a bounded regular open domain of $\mathbb{R}^N$, $N \geq 2$ and $f(x, t)$ satisfies Conditions ($F1$) and ($F2$), then we have:

(i) If $0 < \ell < \lambda_1$ and (F3) holds, then there is no positive solution to problem (1.1)
(ii) If $\ell > \lambda_1$, then the problem (1.1) has a positive solution
(iii) If $\ell = \lambda_1$ and (F3) holds, then problem (1.1) has a positive solution $u\in H_0^1(\Omega)$ if and only if there

exists a constant $c > 0$ such that $u = c\varphi_1$ and $f(x, u) = \lambda_1 u$ *a.e.* in Ω

The next result treat the case $\ell = +\infty$. We prove the existence of a positive solution for the problem in the case of subcritical nonlinearities. More precisely we have.

Theorem 2.2

Let Ω be a bounded regular open domain of $\mathbb{R}^N$, $N \geq 2$ and ($F1$), ($F2$), ($F3$) hold, $\ell = +\infty$ and $f(x, t)$ is subcritical that is $\lim_{t\to+\infty}\frac{f(x,t)}{t^{r-1}} = 0$ uniformly in $x\in\Omega$ for some $r \in \left(2, \frac{2N}{N-2}\right)$ if $N > 2$ or $r\in(2, +\infty)$ if $N = 2$, then the problem (1:1) has a positive solution.

Our method is the variational one. We use the Mountain pass Theorem and we prove the compactness condition without use the (AR) condition introduced by Ambrosetti and Rabionovitz.

Preliminaries and Some Lemmas

Lemma 1

($H_0^1(\Omega)$, $\|.\|$) is a Hilbert space and the norm $\|u\|$ is equivalent to the norm $\|u\|_{H_0^1(\Omega)}$.

Proof

Since $\left(H_0^1(\Omega), \|.\|_{H_0^1(\Omega)}\right)$ is a Banach space where:

$$\|u\|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2\, dx\right)^{\frac{1}{2}}$$

and the function $a(x)$ satisfies the condition (A) so the norm $\|.\|$ is equivalent to the norm $\|.\|_{H_0^1(\Omega)}$ and then $\left(H_0^1(\Omega)\|.\|\right)$ is a Banach space.

Proposition 3.1

Suppose that the function f satisfies ($F1$) and ($F2$), then the following results hold:

(i) There exist ρ, $\beta > 0$ such that $I(u) \geq \beta$ for all $u \in H_0^1(\Omega)$ with $\|.\| = \rho$.

(ii) $I(t\varphi_1) \to -\infty$ as $t \to +\infty$, if $\lambda_1 < \ell < \infty$.

Proof

(i) For any $\varepsilon > 0$, there exist $A = A(\varepsilon) \geq 0$ and $t_0 \geq 1$ such that for all $t \geq t_0$, $f(x, t) \leq 2At$. For $q \geq 1$, we get $f(x, t) \leq 2At^q$ and then:

$$F(x,t) \leq \frac{1}{2}\left(\| p(x)\|_\infty + \varepsilon\right)t^2 + A|t|^{q+1}, \tag{3.1}$$

for all $(x, t) \in \Omega \times \mathbb{R}$.

Let:

$$q^* = \begin{cases} \dfrac{2N}{N-2} & if\ \ N > 2 \\ +\infty & if\ \ N = 2. \end{cases}$$

If we choose q such that $2 < q + 1 < q^*$, then by the Sobolev embedding theorem we obtain $\|u\|_{q+1}^{q+1} \leq C\|u\|^{q+1}$. So, we get:

$$\begin{aligned} I(u) &\geq \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\| p(x)\|_\infty + \varepsilon\right)\|u\|_2^2 - A\|u\|_{q+1}^{q+1} \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\| p(x)\|_\infty + \varepsilon\right)\|u\|_2^2 - AC\|u\|^{q+1}. \end{aligned}$$

Because of the characterization of λ_1, we have:

$$I(u) \geq \frac{1}{2}\left(1 - \frac{\| p(x)\|_\infty + \varepsilon}{\lambda_1}\right)\|u\|^2 - AC\|u\|^{q+1}.$$

With the hypothesis ($F2$) in mind, we can choose $\varepsilon > 0$ such that $\|p(x)\|_\infty + \varepsilon < \lambda_1$ and then we can choose $\|u\| = \rho$ small enough in order to have $I(u) \geq \beta$ for a given $\beta > 0$ sufficiently small.

(ii) Suppose that $\lambda_1 < \ell < +\infty$, for $t > 0$, we have:

$$I(t\varphi_1) = \frac{t^2}{2}\int_\Omega a(x)\left|\nabla\varphi_1\right|^2 - \int_\Omega F(x, t\varphi_1)dx,$$

Since φ_1 is an eigenfunction associated to λ_1, we get:

$$I(t\varphi_1) = \frac{t^2}{2}\lambda_1\int_\Omega \varphi_1^2 dx - \int_\Omega F(x, t\varphi_1)dx. \tag{3.2}$$

By the condition ($F2$) and the asymptotically linearity of $f(x, t)$, we have:

$$\lim_{t\to\infty}\frac{F(x,t)}{t^2} = \frac{\ell}{2}.$$

So, by Fatou's Lemma we obtain:

$$\begin{aligned} \lim_{t\to\infty}\frac{I(t\varphi_1)}{t^2} &\leq \frac{1}{2}\lambda_1\int_\Omega \varphi_1^2 dx - \int_\Omega \lim_{t\to\infty}\frac{F(x,t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2 dx \\ &\leq \frac{1}{2}(\lambda_1 - \ell)\int_\Omega \varphi_1^2 dx. \end{aligned}$$

Since λ_1-$\ell < 0$ and φ_1 is positive, the Proposition 3.1 follows.

As (3.1), for all $\varepsilon > 0$, there exists $B = B(\varepsilon)$ such that:

$$f(x,t) \le \left(\|p(x)\|_\infty + \varepsilon\right)|t| + B|t|,$$

and so, for a fixed small ε, there exists $B = B(\lambda_1)$ such that:

$$f(x,t) \le \lambda_1 |t| + B|t|, \tag{3.3}$$

for all $t \in \mathbb{R}$.

Proposition 3.2

Suppose that the function f satisfies ($F1$), ($F2$) and ($F3$) with $\ell = +\infty$, then the following results hold:

(i) There exist ρ, $\beta > 0$ such that $I(u) \ge \beta$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$

(ii) $I(t\varphi_1) \to -\infty$ as $t \to +\infty$

Proof

(i) Since f is supposed subcritical, that is:

$$\lim_{t\to+\infty} \frac{f(x,t)}{t^{r-1}} = 0 \ \ for\, some\ r \in (2, q^*)$$

and for any $\varepsilon > 0$, there exist $A = A(\varepsilon) \ge 0$ and $t_0 \ge 1$ such that for all $t \ge t_0$, $f(x, t) \le rAt^{r-1}$. Then:

$$F(x,t) \le \frac{1}{2}\left(\|p(x)\|_\infty + \varepsilon\right)t^2 + A|t|^r, \tag{3.4}$$

for all $(x, t) \in \Omega \times \mathbb{R}$.

Since $2 < r < q^*$, by Sobolev embedding theorem we have $\|u\|_r^r \le C\|u\|^r$ and then:

$$\begin{aligned} I(u) &\ge \frac{1}{2}\|u\|^2 - \int_\Omega F(x,u)\,dx \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\|p(x)\|_\infty + \varepsilon\right)\|u\|_2^2 - A\|u\|_r^r \\ &\ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\left(\|p(x)\|_\infty + \varepsilon\right)\|u\|_2^2 - AC\|u\|^r. \end{aligned}$$

So:

$$I(u) \ge \frac{1}{2}\left(1 - \frac{\|p(x)\|_\infty + \varepsilon}{\lambda_1}\right)\|u\|^2 - AC\|u\|^r.$$

As in the proof of Proposition 3.1, we can choose $\varepsilon > 0$ such that $\|p(x)\|_\infty + \varepsilon < \lambda_1$ and then we can choose $\|u\| = \rho$ small enough in order to have $I(u) \ge \beta$ for a given $\beta > 0$ sufficiently small and then the first geometric property is satisfied.

(ii) This second geometric property is essentially due to (Zhou, 2002). For the sake of completeness, we give here a sketch of its proof. The positive function φ_1 is in $C(\Omega)$ (by standard regularity result) so there exist $\Omega_0 \subset \overline{\Omega_0} \subset \Omega$ and $\alpha > 0$ such that $\varphi_1(x) \ge \alpha > 0$ for all $x \in \Omega_0$. From the condition ($F3$), we have:

$$0 \le 2F(x,t) \le tf(x,t), \tag{3.5}$$

and then the function $\frac{F(x,t)}{t^2}$ is nondecreasing with respect to $t > 0$ for a.e. $x \in \Omega$ and $\lim_{t\to+\infty} \frac{F(x,t)}{t^2} = +\infty$.

So, for all $x \in \Omega_0$ and t > 0:

$$\frac{F(x,t\varphi_1(x))}{t^2\varphi_1^2(x)} = \frac{F(x,t\alpha)}{t^2\alpha^2} \to +\infty \ as\ t \to +\infty. \tag{3.6}$$

For all $K > 0$, there exist t_0 such that for all $t \ge t_0$ and $x \in \Omega_0$, we have $\frac{F(x,t\varphi_1(x))}{t^2\varphi_1^2(x)} \ge K$:

$$\begin{aligned} \frac{I(t\varphi_1)}{t^2} &\le \frac{1}{2}\int_\Omega a(x)|\varphi_1|^2\,dx - \int_\Omega \frac{F(x,t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2\,dx \\ &\le \frac{1}{2}\lambda_1 \int_\Omega \varphi_1^2\,dx - K\int_{\Omega_0} \varphi_1^2\,dx \\ &\le \frac{1}{2}\lambda_1 \int_\Omega \varphi_1^2\,dx - K|\Omega_0|\alpha. \end{aligned}$$

For a good $K > 0$, we obtain:

$$\frac{I(t\varphi_1)}{t^2} \le \frac{1}{2}\lambda_1 \int_\Omega \varphi_1^2\,dx - K|\Omega_0|\alpha < 0$$

and so $I(t\varphi_1) \to -\infty$ as $t \to +\infty$.

Lemma 2

Suppose that (v_n) is a convergent sequence to v in $L^p(\Omega)$, for some $1 \le p < +\infty$, then (v_n^+) converges to v^+ in $L^p(\Omega)$, where $v_n^+ = \max(0, v_n)$ and $v^+ = \max(0, v)$.

Proof

We have:

$$v_n^+ = \frac{v_n + |v_n|}{2} \ and\ v^+ = \frac{v + |v|}{2}$$

then:

$$\begin{aligned}
\|v_n^+ - v^+\|_p^p &= \int_\Omega |v_n^+ - v^+|^p dx \\
&= \frac{1}{2^p}\int_\Omega \left|(v_n - v) + (|v_n| - |v|)\right|^p dx \\
&\le \frac{1}{2^p}\int_\Omega \left(|v_n - v| + \left||v_n| - |v|\right|\right)^p dx \\
&\le \frac{1}{2^p}\int_\Omega \left(|v_n - v| + |v_n - v|\right)^p dx \\
&\le \frac{1}{2^p}\int_\Omega \left(2|v_n - v|\right)^p dx \\
&\le \|v_n - v\|_p^p.
\end{aligned}$$

So, $v_n^+ \to v^+$ as $n \to +\infty$, in $L^p(\Omega)$.

In similar way as in (Stuart and Zhou, 1999; Zhou, 2002), we have the following result.

Lemma 3

If $< I'(u_n), u_n > \to 0$ and ($F3$) holds, then there exists a subsequence of (u_n), still denote by (u_n), such that $I(tu_n) \le \frac{1+t^2}{2n} + I(u_n)$, for all $t > 0$.

Proof of the Main Results

Proof of the Theorem 2.1

(i) Suppose that $0 < \ell < \lambda_1$ and the function f satisfies the conditions ($F1$)-($F3$). We will prove that the problem (1.1) does not have a positive solution by contradiction.

Suppose that $u \in H_0^1(\Omega)$ is a positive solution of the problem (1.1), so u satisfies the equation (1.3) for all $\varphi \in H_0^1(\Omega)$, in particular for $\varphi = u$, we get:

$$\int_\Omega a(x)|\nabla u|^2 dx = \int_\Omega f(x,u)u dx \le \int_\Omega \ell u^2 dx \tag{4.1}$$

from ($F1$), ($F2$) and ($F3$).

By definition of λ_1, we get $\lambda_1 \le \ell$ and this contradicts the fact that $\ell < \lambda_1$.

(iii) If $\ell = \lambda_1$. Suppose that u is a positive solution for the problem (1.1) and take $\varphi_1 \in H_0^1(\Omega)$ as test function in (1.3), we get:

$$\int_\Omega a(x)\nabla u.\nabla\varphi_1 dx = \int_\Omega f(x,u)\varphi_1 dx. \tag{4.2}$$

Now, we consider the equation verified by φ_1 and take u as test function, we obtain:

$$\int_\Omega a(x)\nabla u.\nabla\varphi_1\, dx = \ell\int_\Omega u\varphi_1 dx \tag{4.3}$$

and so $\int_\Omega (f(x,u) - \ell u)\varphi_1\, dx = 0$. Since φ_1 positive and the function $f(x, t)$ satisfies ($F2$) and ($F3$), we conclude that $f(x, u) = \ell u$ a.e. in Ω. But $\ell = \lambda_1$ and then u is an eigenfunction associated to the simple eigenvalue λ_1, so $u = c\varphi_1$ for some constant $c > 0$.

Conversely, if for some constant $c > 0$, $u = c\varphi_1$ and $f(x, u) = \lambda_1 u$, then:

$$\begin{aligned}
-div(a(x)\nabla u) &= -c\, div(a(x)\nabla\varphi_1) \\
&= c\lambda_1\varphi_1 \\
&= \lambda_1 u \\
&= f(x,u).
\end{aligned}$$

Also the boundary conditions are satisfied and then u is a positive solution for the problem (1.1).

(ii) Suppose that $0 < \ell < \lambda_1$ and the function f satisfies the conditions ($F1$)-($F2$).

The space $(H_0^1(\Omega), \|.\|)$ is a Banach space and the functional I, given by (1.4), is C^1 and satisfies $I(0) = 0$.

By Proposition 3.1, there exist $\beta, \rho > 0$ such that $I(u) \ge \beta$ for all $u \in H_0^1(\Omega)$ with $\|u\| = \rho$ and there exists $u_1 \in H_0^1(\Omega)$ such that $\|u_1\| > \rho$ and $I(u_1) < 0$.

In order to prove the compactness condition, let (u_n) be a Palais Smale sequence at level $d \in \mathbb{R}$, that is:

$$I(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F(x,u_n)dx \to d \quad as\ n \to +\infty \tag{4.4}$$

and:

$$\|I'(u_n)\|_* \to 0 \quad as\ n \to +\infty. \tag{4.5}$$

and we have to prove that (u_n) contains a convergent subsequence in $H_0^1(\Omega)$. For this, it suffices to prove that (u_n) is bounded in $H_0^1(\Omega)$. Indeed, up to a subsequence we obtain:

$$\begin{aligned}
&u_n \to u \ weakly\, in\, H_0^1(\Omega) \\
&u_n \to u \ stongly\, in\, L^2(\Omega) \\
&u_n \to u \ a.e\ in(\Omega).
\end{aligned}$$

Moreover, by the trace theorem:

$$u = 0\ on\ \partial\Omega.$$

Clearly (4.5) implies that:

$$< I'(u_n), u_n >= \|u_n\|^2 - \int_\Omega f(x,u_n)u_n \to 0 \tag{4.6}$$

and:

$$\int_\Omega a(x)\nabla u_n.\nabla\varphi - \int_\Omega f(x,u_n)\varphi \to 0 \ \ for\,all\,\varphi\in H_0^1(\Omega), \quad (4.7)$$

that is:

$$-div\big(a(x)\nabla u_n\big) - f(x,u_n) \to 0 \ in\ \mathcal{D}'(\Omega). \quad (4.8)$$

Note that by ($F2$), $f(x, u_n) \to f(x, u)$ in $L^2(\Omega)$ and in view of the $L^2(\Omega)$-convergence of (u_n):

$$\int_\Omega f(x,u_n)u_n \to \int_\Omega f(x,u)u.$$

and then (4.8) gives:

$$-div\big(a(x)\nabla u_n\big) \to f(x,u)\ in\ \mathcal{D}'(\Omega) \quad (4.9)$$

and so:

$$-div\big(a(x)\nabla u\big) = f(x,u)\ in\ \Omega. \quad (4.10)$$

From (4.10) and by taking u as test function, we get:

$$\|u\|^2 - \int_\Omega f(x,u)u = 0. \quad (4.11)$$

Using (4.6) and (4.10), we have $\|u_n\|^2 \to \|u\|^2$ which insures that $u_n \to u$ in $H_0^1(\Omega)$.

To finish the proof, we have to prove that the (PS)-sequence (u_n) is bounded in $H_0^1(\Omega)$. From (4.4) and (3.2), we have only to prove that (u_n) is bounded in $L^2(\Omega)$. We suppose, by contradiction, that $\|u_n\|_2 \to +\infty$ and let:

$$w_n = \frac{u_n}{\|u_n\|_2}, k_n = \|u_n\|_2. \quad (4.12)$$

We have:

$$\lim_{n\to\infty}\frac{I(u_n)}{k_n^2} = \lim_{n\to\infty}\left[\frac{1}{2}\|w_n\|_d^2 - \frac{1}{k_n^2}\int_\Omega F(x,u_n)\right] = 0. \quad (4.13)$$

By (3.2), $(1/k_n^2)\int_\Omega F(x,u_n)$ is bounded and so (4.13) shows that w_n is bounded in $H_0^1(\Omega)$.

Let $w\in H_0^1(\Omega)$ be such that:

$$w_n \to w\ weakly\, in\, H_0^1(\Omega)$$
$$w_n \to w\ strongly\, in\, L^2(\Omega)$$
$$w_n \to w\ a.e\ in(\Omega).$$

We claim that:

$$-div\big(a(x)\nabla w\big) = \ell w^+\ \ in\Omega. \quad (4.14)$$

For the proof of the claim, we divide (4.7) by k_n, then we get:

$$\int_\Omega a(x)\nabla w_n.\nabla\varphi - \frac{1}{k_n}\int_\Omega f(x,u_n)\varphi \to 0\ \ for\,all\,\varphi\in H_0^1(\Omega). \quad (4.15)$$

Also, we have:

$$\int_\Omega a(x)\nabla w_n.\nabla\varphi \to \int_\Omega a(x)\nabla w.\nabla\varphi. \quad (4.16)$$

If we prove that $\frac{1}{k_n}f(x,u_n)$ converges (up to subsequence) to ℓw^+ in $L^2(\Omega)$, then (4.14) follows.

By ($F2$), $\frac{1}{k_n}f(x,k_nw_n)$ converges to ℓw^+ in the set:

$$\{x\in\Omega / w_n(x) \to w(x) and\ w(x)\neq 0\}.$$

If $w_n(x) \to w(x)$ and $w(x) = 0$, from (3.2) we deduce that $\frac{1}{k_n}f(x,k_nw_n)$ converges to zero. Thus $\frac{1}{k_n}f(x,u_n)$ converges to ℓw^+ a.e. in Ω.

Since $w_n \to w$ in $L^2(\Omega)$, by Theorem IV.9 (Brezis, 2010), w_n is dominated in $L^2(\Omega)$ (up to subsequence) and then $\frac{1}{k_n}f(x,k_nw_n)$ is dominated and then we conclude that $\frac{1}{k_n}f(x,k_nw_n)$ converges to ℓw^+ in $L^2(\Omega)$ and so the propriety 4.14 is proved and we get:

$$\begin{cases} -div\big(a(x)\nabla w\big) = \ell w^+\ in\ \Omega \\ \qquad\qquad w = 0 \quad on\ \partial\Omega. \end{cases} \quad (4.17)$$

By the maximum principle, $w > 0$ and then w is a solution of:

$$\begin{cases} -div\big(a(x)\nabla w\big) = \ell w\ in\ \Omega \\ \qquad\qquad w = 0 \quad on\ \partial\Omega \\ \quad \int_\Omega w^2 = 1 \end{cases} \quad (4.18)$$

and then $w = \varphi_1$ and $\ell = \lambda_1$, which contradicts the fact that $\lambda_1 < \ell < \infty$.

Proof of the Theorem 2.2

In this case $\ell = +\infty$. To prove the existence of critical point of the functional I, let (u_n) a Palais Smale sequence satisfying (4.4) and (4.7). Following the same procedures as in the proof of Theorem 2.1 (*ii*), we have to prove that the sequence (u_n) is bounded in $H_0^1(\Omega)$. By contradiction, suppose that (u_n) is not bounded in $H_0^1(\Omega)$. Let c a positive real number and consider:

$$k_n = \frac{1}{c\|u_n\|},\ w_n = k_n u_n. \tag{4.19}$$

w_n is bounded in $H_0^1(\Omega)$, then there exists $w \in H_0^1(\Omega)$ such that, up to subsequence:

$$\begin{aligned} &w_n \to w \textit{ weakly in } H_0^1(\Omega) \\ &w_n \to w \textit{ strongly in } L^2(\Omega), \\ &w_n \to w \textit{ a.e in}(\Omega). \end{aligned}$$

By Lemma 2, we have:

$$w_n^+ \to w^+ \textit{ strongly in } L^2(\Omega),$$

and:

$$w_n^+(x) \to w^+(x) \textit{ a.e. in } \Omega.$$

We claim that:

$$w^+(x) = 0 \textit{ a.e. in } \Omega. \tag{4.20}$$

Indeed, let $\Omega_1 = \{x \in \Omega,\ w^+(x) = 0\}$ and $\Omega_2 = \{x \in \Omega;\ w^+(x) > 0\}$.

By (4.19), $u_n^+(x) \to +\infty$ a.e. in Ω_2 so by (*F*2), for a given $K > 0$ and n large enough we have:

$$\frac{f\left(x, u_n^+(x)\right)}{u_n^+(x)}\left(w_n^+(x)\right)^2 \geq K\left(w^+(x)\right)^2 \ \textit{ for } n \geq n_0. \tag{4.21}$$

From (4.6) and (4.12) we get:

$$\begin{aligned} \frac{1}{c^2} &= \lim_{n\to+\infty}\|w_n\|_d^2 = \lim_{n\to+\infty}\int_\Omega \frac{f(x,u_n)}{u_n}(w_n)^2 \\ &\geq \lim_{n\to+\infty}\int_{\Omega_2} \frac{f(x,u_n^+)}{u_n^+}(w_n^+)^2 \\ &\geq \int_{\Omega_2}\lim_{n\to+\infty} \frac{f(x,u_n^+)}{u_n^+}(w_n^+)^2 \\ &\geq K\int_{\Omega_2}(w^+)^2 \end{aligned}$$

and K can be chosen large enough. So $|\Omega_2| = 0$ and then $w^+ \equiv 0$ in Ω.

Now since $w^+ \equiv 0$, $\lim_{n\to+\infty}\int_\Omega F\left(x, w_n^+(x)\right)dx = 0$ and hence:

$$\lim_{n\to+\infty} I(w_n) = \frac{1}{2c^2}. \tag{4.22}$$

If we apply Lemma 3, we have up to subsequence:

$$I(w_n) = I(k_n u_n) \leq \frac{1}{2n}\left(1 + k_n^2\right) + I(u_n), \tag{4.23}$$

$k_n = \dfrac{1}{c\|u_n\|} \to 0$ as n tends to $+\infty$, then we have:

$$\frac{1}{2c^2} \leq \lim_{n\to+\infty} I(u_n) \tag{4.24}$$

this is for any $c > 0$ which contradict the hypothesis (4.4). So, the proof of the Theorem 2.2 is finished.

Conclusion

We have proved the existence of positive solution for a family of nonlinear elliptic problems by using Mountain Pass Theorem and critical point theory without addition condition on the nonlinearities of type Ambrosetti and Rabionovitz condition (*AR*) or one of its refinements (Ambrosetti and Rabinowitz, 1973). A general condition on the weight function $a(x)$ is considered for the problem of equivalence of norms. We can also treat and discuss the same problem with a singular weight function and asymptotically nonlinearities. May be we can use what we call weighted Lebesgue and Sobolev spaces.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that there is no other authors to read and approved the manuscript and no ethical issues involved.

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