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Dynamics and Stability of q-Fractional Order Pantograph Equations With Nonlocal Condition

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Abstract: This paper deals with some existence and Ulam-Hyers-q-Mittag-Leffler stability results for q-fractional order pantograph equation. An application is made of a Darbo's fixed point theorem for the existence of solutions.

Keywords: Fractional q-Fractional Order Pantograph Equation, Nonlocal Condition, Existence, Fixed Point, Ulam-Hyers-q-Mittag-Leffler Stability

Introduction

Early works about q -difference calculus or quantum calculus initially read in Jackson (1908; 1910). The difference equations are widely used in mathematical physical problems, sampling theory of signal analysis, dynamical system and quantum models and heat and wave equations. Recently, some researchers have noticed their attention to discrimination research of the fractional q-difference calculus, we refer readers to (Agarwal *et al*., 2014; Liang and Zhang, 2012; Al-Yami, 2016; Stankovic *et al*., 2009). For a long time, in many areas the fractional differential equations are very popular. For example, engineers and scientists have developed new methods that include fractional equations; we refer to (Hilfer, 1999; Podlubny, 1999). Since the beginning of the last decade, the fractional q-differential equations have become an important mathematical tool.

The pantograph equations have been widely studied (Balachandran *et al*., 2013; Liu and Li, 2004) and references therein)since they can be utilized to depict many phenomena that arise in electro dynamics, probability, quantum mechanics, dynamical systems and number theory. Recently, fractional pantograph differential equations have been considered by many researchers. One of the motivating topics in this area is the research of the existence of solutions by fixed point theorems, we refer to (Balachandran *et al*., 2013).

The Ulam stability of functional equation, which was Ulam founded for a speech to a conference at the University of Wisconsin in 1940, is one of the important subjects in the mathematical analysis area. The finding of Ulam stability plays a vital role in regard to this subject. For detailed study on the progress of Ulam type (U-H) stability, readers refer to (Andras and Kolumban, 2013; Jung, 2004; Muniyappan and Rajan, 2015) and the references therein. The credit of solving this problem partially goes to Hyers. To study U-H stability of fractional differential equations, different researchers studied their works with different methods, see (Ibrahim, 2012; Wang *et al*., 2011; Wang and Zhou, 2012; Wang and Zhang, 2014). Koca (2015) proved local asymptotics stability of q-fractional nonlinear dynamics systems. Inspired by the above discussion, we initiate the existence and U-H-q-Mittag-Leffler stability for q-fractional pantograph equations.

Consider the following system represented by the q-fractional order pantograph equation with nonlocal condition of the form:

$$\begin{aligned} &{}^{c}D_q^{\alpha}x(t) = F\big(t, x(t), x(\lambda t)\big), t \in \mathscr{J} := [0,T] \\ &x(0) + g(x) = x_0 \end{aligned} \tag{1}$$

where, ${}^{c}D_q^{\alpha}$ is the Caputo fractional q-derivative, $q\in$ (0,1). Let $0 < \alpha < 1$, $0 < \lambda < 1$ and $F : \mathscr{J} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$, $g : \mathscr{C}(\mathscr{J}, \mathfrak{X}) \to \mathfrak{X}$ are given continuous functions.

Observing that system (1) is equivalent to the following nonlinear integral equation:

$$\begin{aligned} &x(t) = x_0 - g(x) \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} F\big(s, x(s), x(\lambda s)\big) d_q s \end{aligned} \tag{2}$$

Let $\mathscr{C}(\mathscr{J}, \mathfrak{X})$ be the Banach space of continuous function $x(t)$ with $x(t) \in \mathfrak{X}$ for $t \in \mathscr{J}$ and $\|x\|\ \mathscr{C}(\mathscr{J}, \mathfrak{X}) = \max_{t\in\mathscr{J}}\|x(t)\|$.

In passing, we note that the application of nonlinear condition $x(0) + g(x) = x_0$ in physical problems yeilds better effect than the initial condition $x(0) = x_0$ (Bashir and Sivasundaram, 2008).

The outline of the paper is as follows. In section 2, we give some basic definitions and results con-cerning the fractional q-calculus. In section 3, we present our main results by fixed point theorems. Stability analysis is discussed in section 4.

Prerequisites

For detailed study on q-fractional calculus, one can refer to (Al-Yami, 2016; Stankovic *et al*., 2009).

Let $q \in (0, 1)$ and define:

$$[a]_q = \frac{q^a - 1}{q - 1} = q^{a-1} + \cdots + 1, a \in \mathbb{R}$$

The q-analogue of the Pochhammer symbol was presented as follow:

$$(a;q)_0 = 1, (a;q)_n = \prod_{k=0}^{n-1}(1 - aq^k), a \in \mathbb{R}, n \in \mathbb{N} \cup \{\infty\}$$

In general, if $\alpha \in \mathbb{R}$ thereafter:

$$(a;q)_\infty = \prod_{i=0}^{\infty}(1 - aq^i), (a;q)_\alpha \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}$$

The q-gamma function is defined by:

$$\Gamma_q(x) = (q;q)_{x-1}(1-q)^{1-x}, x \in \mathbb{R}| \{0, -1, -2, ...\}, 0 < q < 1$$

and satisfies $\Gamma_q(x + 1) = [x]_q\Gamma_q(x)$.

The q-derivative of a function F(x) is here defined by:

$$D_q F(x) = \frac{d_q F(x)}{d_q x} = \frac{F(qx) - F(x)}{(q-1)x}$$

and:

$$D_q^n F(x) = \begin{cases} F(x) & if n = 0 \\ D_q D_q^{n-1} F(x) & if n \in \mathbb{N} \end{cases}$$

The q-integral of a function F defined in the interval [0, T] is provided by:

$$\int_0^x F(t) d_q t = x(1-q)\sum_{n=0}^{\infty} F(xq^n) q^n, 0 \le |q| < 1, \quad x \in [0, b]$$

now, it can be defined an operator I_q^n, as follows:

$$(I_q^0 F)(x) = F(x) \text{ and } (I_q^n F)(x) = I_q(I_q^{n-1}F)(x), \ n \in N$$

We can point the basic formula which will be used at a later time:

$${}_sD_q t^\alpha (s/t;q)_\alpha = -[\alpha]_q t^{\alpha-1}(qs/t;q)_{\alpha-1}$$

where, ${}_sD_q$ denotes the q-derivative with respect to variable s.

Definition 2.1. (Al-Yami, 2016)

Let $\alpha \le 0$ and F be a function defined on [0, T]. The fractional q-integral of the Riemann-liouville type is $(I_q^0 F)(x) = F(x)$ and:

$$(I_q^\alpha F)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^x (qt/x;q)_{\alpha-1} F(t) d_q t, \quad \alpha \in \mathbb{R}^+, x \in [0, T]$$

Definition 2.2. (Al-Yami, 2016)

The fractional q-derivative of the Caputo type of order $\alpha > 0$ is defined by:

$$({}_cD_q^\alpha F)(x) = (I_q^{[\alpha]-\alpha} D_q^{[\alpha]} F)(x)$$

where, $[\alpha]$ is the smallest integer greater than or equal to α.

Definition 2.3

The fractional q-derivative of the Riemann-Lioville type of order $\alpha \ge 0$ is defined by $D_q^0 f(x) = f(x)$ and:

$$(D_q^0 F)(x) = (D_q^{\lceil\alpha\rceil} I_q^{m-\lceil\alpha\rceil} F)(x) \alpha > 0$$

Lemma 2.4. (Al-Yami, 2016)

Let $x > 0$ and $\alpha \in \mathbb{R}^+/\mathbb{N}$. Then, the following equality holds:

$$(I_q^\alpha\ {}_cD_q^\alpha F)(x) = F(x) - \sum_{k=0}^{\lceil\alpha\rceil} \frac{x^k}{\Gamma_q(k+1)} (D_q^k F)(0)$$

Denition 2.5. (Hassan, 2016)

The q-Mittag-Leffler-function defined as:

$$e_{\alpha,\mu}(z;q) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma_q(\alpha k + \mu)}, |z| < (1-q)^\alpha$$

when $\mu = 1$ we simply use $e_\alpha(z;q) := e_{\alpha,1}(z;q)$.

Remark 2.6

The q-Mittag-Leffler function will tend to the classical one when $q \to 1$.

Theorem 2.7. (*Darbo's Fixed Point Theorem (Lakshmikantham, 1994),p.no.21*)

Let K be a bounded, closed convex set of a Banach space $\mathscr{X}$. Suppose that T and S are two mappings from K to $\mathscr{X}$ satisfying:

- $Tx + Sy \in K$ for any $x, y \in K$
- T is a contraction mapping
- S is a completely continuous on K

Then $T + S$ has atleast a fixxed point on K.

Main Results

Let us list some hypotheses to prove our existence results:

(A1) $F : \mathscr{J} \times \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ is continuous function

(A2) There exists a positive constant $L > 0$ such that:

$$|F(t,x,u) - F(t,y,\upsilon)| \le L(|x-y| + |u-\upsilon|)$$

for $t \in \mathscr{J}, u,\upsilon,x,y \in \mathscr{X}$

(A3) $g : \mathscr{C}(\mathscr{J}, \mathscr{X}) \to \mathscr{X}$ is continuous function and there exists a constant $b > 0$, such that:

$$|g(x) - g(y)| \le b|x-y|, \textit{ for all } x,y \in \mathscr{C}(\mathscr{J},\mathscr{X})$$

(A4) There exists a function $\mu \in L^1(\mathscr{J})$ such that:

$$|F(t,x,y)| \le \mu(t) \textit{ for all } t \in \mathscr{J}, x,y \in \mathscr{X}$$

We are now ready to present our results. The existence results are based on Darbo's fixed point theorem.

Theorem 3.1

Assume (*A1*), (*A3*) with $b < 1$ and (*A4*) hold. Then, system (1) has at least one fixed point on $\mathscr{J}$.

Proof

Let P and Q the two operators defined on B_r by:

$$(Px)(t) := \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} F(s,x(s),x(\lambda s))\,d_q s,$$

$$(Qx)(t) := x_0 - g(x)$$

respectively. Note that if $x, y \in B_r$, where, $B_r := \{x \in \mathscr{C}(\mathscr{J}, \mathscr{X}) : |x| \le r\}$.

Set $G = \max_{x \in \mathscr{C}(\mathscr{J},\mathscr{X})} |g(x)|$, then $Px + Qy \in B_r$.

Indeed it is easy to check the inequality:

$$\begin{aligned}
|Px+Qy| &= \left| x_0 - g(y) + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} F(s,x(s),x(\lambda s))\,d_q s \right| \\
&\le |x_0| + |g(y)| + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \\
&\int_0^t |(qs/t;q)_{\alpha-1} F(s,x(s),x(\lambda s))|\,d_q s \\
&\le |x_0| + G + \frac{|\mu| t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1}\,d_q s \\
&\le \|x_0\| + G + \frac{\|\mu\|_{L^1} T^\alpha}{\Gamma_q(\alpha+1)} \le r
\end{aligned}$$

Thus:

$$Px + Qy \in B_r$$

By (A3), it is also clear that Q is a contraction mapping. Produced from continuity of x, the operator $(Px)(t)$ is continuous in accordance with ($A1$). Also we observe that:

$$\begin{aligned}
|(Px)(t)| &\le \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} |F(s,x(s),x(\lambda s))|\,d_q s \\
&\le \frac{\|\mu\|_{L^1} T^\alpha}{\Gamma_q(\alpha+1)}
\end{aligned}$$

Then P is uniformly bounded on B_r.

Now let's prove that $(Px)(t)$ is equicontinuous.

Let $t_1, t_2 \in J$, $t_2 \le t_1$ and $x \in B_r$. Using the fact F is bounded on the compact set $\mathscr{J} \times B_r$ (thus $\max_{(t,x,y)\in \mathscr{J} \times B_r} |F(t, x, y)| := C_0 < \infty$).

We will get:

$$\begin{aligned}
&|(Px)(t_1) - (Px)(t_2)| \\
&= \left| \begin{array}{l} \dfrac{t_1^{\alpha-1}}{\Gamma_q(\alpha)} \displaystyle\int_0^{t_1} (qs/t_1;q)_{\alpha-1} F(s,x(s),x(\lambda s))\,d_q s \\ -\dfrac{t_2^{\alpha-1}}{\Gamma_q(\alpha)} \displaystyle\int_0^{t_2} (qs/t_2;q)_{\alpha-1} F(s,x(s),x(\lambda s))\,d_q s \end{array} \right| \\
&\le \frac{1}{\Gamma_q(\alpha)} \left[\begin{array}{l} t_1^{\alpha-1} \displaystyle\int_{t_2}^{t_1} (qs/t_1;q)_{\alpha-1} |F(s,x(s),x(\lambda s))|\,d_q s \\ + \displaystyle\int_0^{t_2} \left(t_2^{\alpha-1} (qs/t_2;q)_{\alpha-1} - t_1^{\alpha-1} (qs/t_1;q)_{\alpha-1} \right) \\ |F(s,x(s),x(\lambda s))|\,d_q s \end{array} \right] \\
&\le \frac{C_0}{\Gamma_q(\alpha)} \left[\begin{array}{l} t_1^{\alpha-1} \displaystyle\int_{t_2}^{t_1} (qs/t_1;q)_{\alpha-1}\,d_q s \\ + \displaystyle\int_0^{t_2} \left(t_2^{\alpha-1} (qs/t_2;q)_{\alpha-1} - t_1^{\alpha-1} (qs/t_1;q)_{\alpha-1} \right) d_q s \end{array} \right]
\end{aligned}$$

which is autonomous of x and head for zero as $t_1 - t_2 \to 0$ consequently P is equicontinuous. Thus, P is relatively compact on B_r. By the Arzela-Ascoli theorem, P is compact. We now conclude the result of the theorem based on the Darbo's fixed point. Thus, the problem (1) has at least one fixed point on $\mathscr{J}$.

Stability Analysis

In this section, we define some basic concepts of U-H-q-Mittag-Leffler stability. We adopt some ideas in (Otrocol and Ilea, 2013).

Definition 4.1

The Equation (1) is *U-H-q*-Mittag-Leffler stable with respect to $e_\alpha(t^\alpha;q)$ if there exists C_{e_α} such that for each $\epsilon > 0$ and for each solution $z \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ of the inequality:

$$\left|{}^cD_q^\alpha z(t) - F(t,z,(t),z(\lambda t))\right| \le \epsilon e_\alpha(t^\alpha;q), \quad t \in \mathscr{J}$$

There exists a solution $x \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ of Equation (1) with:

$$|z(t) - x(t)| \le C e_\alpha \epsilon e_\alpha(t^\alpha;q), \quad t \in \mathscr{J}$$

where, $e_\alpha(t^\alpha;q)$ is the q-Mittag-Leffler function.

Remark 4.2

A function $z \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ is a solution of the inequality:

$$\left|{}^cD_q^\alpha z(t) - F(t,z(t),z(\lambda t))\right| \le \epsilon e_\alpha(t^\alpha;q), \quad t \in \mathscr{J}$$

if and only if there exists a function $h \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ such that:

1. $|h(t)| \le \epsilon e_\alpha(t^\alpha;\upsilon), \quad t \in \mathscr{J}$
2. ${}^cD_q^\alpha z(t) = F(t,z(t),z(\lambda t)) + h(t), \qquad t \in \mathscr{J}$

Lemma 4.3

If a function $z \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ is a solution of the inequality:

$$\left|{}^CD_q^\alpha z(t) - F(t,z)(t),z(\lambda t))\right| \le \epsilon e_\alpha(t^\alpha;q), \quad t \in \mathscr{J}$$

then:

$$\left| z(t) - z_0 + g(z) - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} F(s,z(s),z(\lambda s)) d_q s \right| \le \epsilon e_\alpha(t^\alpha;q)$$

Proof

The proof directly follows from Remark 4.2, we have:

$$\left| z(t) - z_0 + g(z) - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} F(s,z(s),z(\lambda s)) d_q s \right|$$

$$\le \frac{\epsilon}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} e_\alpha(s^\alpha;q) d_q s$$

$$\le \frac{\epsilon}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \sum_{k=0}^{\infty} \frac{s^{k\alpha}}{\Gamma_q(k\alpha+1)} d_q s$$

$$\le \frac{\epsilon}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} \frac{s^{k\alpha}}{\Gamma_q(k\alpha+1)} \int_0^t (t-qs)^{(\alpha-1)} s^{k\alpha} d_q s$$

$$= \epsilon \sum_{k=0}^{\infty} \frac{t^{(k+1)}\alpha}{\Gamma_q((k+1)\alpha+1)}$$

$$= \epsilon \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma_q(n\alpha+1)}$$

$$= \epsilon e_\alpha(t^\alpha;q)$$

Denition 4.4. (Otrocol and Ilea, 2013)

Let $(\mathscr{X}, d)$ be a metric space. *An A*: $\mathscr{X} \to \mathscr{X}$ is a Picard operator if there exists $x^* \in \mathscr{X}$ such that (*i*) $F_A = x^*$ where $F_A = \{x \in \mathscr{X}: A(x) = x\}$ is the fixed point set of A; (*ii*) the sequence $(A^n(x_0))_{n\in\mathrm{N}}$ converges to x^* for all $x_0 \in \mathscr{X}$.

Lemma 4.5. (abstract Gronwall lemma (Otrocol and Ilea, 2013))

Let $(\mathscr{X}, d, \le)$ be an ordered metric space and A: $\mathscr{X} \to \mathscr{X}$ be an increasing Picards operator $\left(F_A = \{x_A^*\}\right)$. Then, for $x \in \mathscr{X}$, $x \le A(x)$ implies $x \ge x_A^*$.

Lemma 4.6. (Henry-Gronwall inequality (Hyers et al., 1998))

Let y,w: $[0, T) \to [0,\infty)$ be continuous function where $T \le \infty$. If w is nondecreasing and there are constants $k \ge 0$ and $0 < \alpha < 1$ such that:

$$y(t) | \le w(t) + k \int_0^t (t-qs)^{(\alpha-1)} y(s) d_q s, \quad t \in [0,T)$$

then:

$$y(t) w(t) + \int_0^t \left(\sum_0^t \frac{(k\Gamma_q(\alpha))^n}{\Gamma_q(n\alpha)} (t-qs)^{(n\alpha-1)} w(s) \right) d_q s, \ t \in [0,T)$$

Remark 4.7

By the hypothesis of Lemma 4.6, let $w(t)$ be a nondecresing function on $[0, T)$. Then we have $y(t) \le w(t) e_\alpha(k\Gamma_q(\alpha) t^\alpha; q)$.

We are now in a position to state and prove our stability results for problem (1). The arguments are based on the Banach contraction principle with respect to Chebyshev norm.

Theorem 4.8

Assume that hypotheses (A1)-(A3) are fulfilled. If:

$$b < \frac{1}{2} \text{ and } L \le \frac{\Gamma_q(\alpha+1)}{4T^\alpha} \tag{3}$$

Then the Equation (1) has a unique solution.

Proof

The operator P: $\mathscr{C}(\mathscr{J}, \mathscr{X}) \to \mathscr{C}(\mathscr{J}, \mathscr{X})$:

$$(Px)(t) = x_0 - g(x) + \frac{1}{\Gamma_q(x)} \int_0^t (t-qs)^{(\alpha-1)} F(s, x(s), x(\lambda s)) d_q s$$

Choose $r \ge 2\left(\|x_0\| + G + \frac{MT^q}{\Gamma_q(\alpha+1)} \right)$ and let $\max_{t\in\mathscr{J}}$ $|F(t, 0, 0)| = M$.

Then we can show that $PB_r \subset B_r$.

So let $x \in B_r$ and set $G = \max_{x\in \mathscr{C}(\mathscr{J},\mathscr{X})}|g(x)|$. Then we get:

$$\begin{aligned}
&|Px(t)| \\
&= \left| x_0 - g(x) + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{(\alpha-1)} F(s, x(s), x(\lambda s)) d_q s \right| \\
&\le |x_0| + G + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t |(qs/t;q)_{\alpha-1} F(s, x(s), x(\lambda s))| d_q s \\
&\le |x_0| + G + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} \\
&\left(\max_{s\in J} |F(s, x(s), x(\lambda s)) - F(s,0,0)| + \max_{s\in J} |F(s,0,0)| \right) d_q s \\
&\le |x_0| + G + \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} L\left[|x(s)| + |x(\lambda s)| \right] d_q s \\
&+ \frac{Mt^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} d_q s \\
&\le |x_0| + G + \frac{2Lrt^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} d_q s + \frac{MT\alpha}{\Gamma_q(\alpha+1)} \\
&\le |x_0| + G + \frac{2L_r T^\alpha}{\Gamma_q(\alpha+1)} + \frac{MT^\alpha}{\Gamma_q(\alpha+1)} \\
&\le \|x_0\|_{C(J,X)} + G + (2Lr + M) \frac{T^\alpha}{\Gamma_q(\alpha+1)} \le r
\end{aligned}$$

By the choice of L and r. Now take $x, y \in \mathscr{C}(\mathscr{J}, \mathscr{X})$. Then we get:

$$\begin{aligned}
&|(Px)(t) - (Py)(t)| \\
&\le |g(x) - g(y)| \\
&+ \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} \left| \begin{matrix} F(s, x(s), x(\lambda s)) \\ -F(s, y(s), y(\lambda s)) \end{matrix} \right| d_q s \\
&\le b|x-y| \\
&+ \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} L \left(\begin{matrix} \max_{s\in\mathscr{J}} |x(s) - y(s)| \\ + \max_{s\in\mathscr{J}} |x(\lambda s) - y(\lambda s)| \end{matrix} \right) d_q s \\
&\le b|x-y| + \frac{2L|x-y|t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t;q)_{\alpha-1} d_q s \\
&\le b|x-y| + \frac{2L|x-y|T^\alpha}{\Gamma_q(\alpha+1)} \\
&\le \left(b + \frac{2LT^\alpha}{\Gamma_q(\alpha+1)} \right) \|x-y\|_{\mathscr{C}(\mathscr{J},\mathscr{X})} = \Omega_{b,L,T,\alpha,q} \|x-y\|_{\mathscr{C}(\mathscr{J},\mathscr{X})}
\end{aligned}$$

Thus:

$$\|(Px) - (Py)\|_{\mathscr{C}(\mathscr{J};\mathscr{X})} \le \Omega_{b,L,T,\alpha,q} \|x-y\|_{\mathscr{C}(\mathscr{J};\mathscr{X})}$$

where, $\Omega_{b,L,T,\alpha,q} := \left(b + \frac{2LT^\alpha}{\Gamma_q(\alpha+1)} \right)$ depends only on the parameters of the problem and since $\Omega_{b,L,T,\alpha,q} < 1$, the result follows in view of the contraction mapping principle due to Chebyshev norm.

Theorem 4.9

If the hypotheses (A1)-(A3) and (3) are satisfied. Then, the Equation (1) is *U*-*H*-*q*-Mittag-Leffler stable.

Proof

Let $\epsilon > 0$ and let $z \in \mathscr{C}(\mathscr{J},\mathscr{X})$ be a function which satisfies the inequality:

$$\left| {}^c D_q^\alpha z(t) - F(t, z(t), z(\lambda t)) \right| \le \epsilon e_\alpha(t^\alpha; q), \text{ for any } t \in \mathscr{J} \tag{4}$$

and let $x \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ be the unique solution of the following q-fractional order pantograph equation:

$$\begin{aligned}
&{}^c D_q^\alpha x(t) = F(t, x(t), x(\lambda t)), \quad t \in \mathscr{J}, 0 < \alpha < 1, \\
&z(0) + g(z) = x_0
\end{aligned}$$

So:

$$x(t) = x_0 - g(x) + \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} F(s, x(s), x(\lambda s)) d_q s$$

By applying Lemma 4.3, we get:

$$\left| \begin{matrix} z(t) - x_0 + g(z) \\ - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} F(s, z(s), z(\lambda s)) d_q s \end{matrix} \right| \leq \epsilon e_\alpha(t^\alpha; q)$$

For each $t \in \mathscr{J}$, we have:

$$\begin{aligned} &|z(t) - x(t)| \\ &\leq \left| \begin{matrix} z(t) - x_0 + g(z) \\ - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} F(s, z(s), z(\lambda s)) d_q s \end{matrix} \right| \\ &+ |g(x) - g(z)| \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \begin{pmatrix} F(s, z(s), z(\lambda s)) \\ -F(s, x(s), x(\lambda s)) \end{pmatrix} d_q s | \end{aligned} \quad (5)$$

$$\begin{aligned} &\leq \epsilon e_\alpha(t^\alpha; q) + |g(z) - g(x)| \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} \\ &|F(s, z(s), z(\lambda s)) - F(s, x(s), x(\lambda s))| d_q s \\ &\leq \epsilon e_\alpha(t^\alpha; q) + b|z(t) - x(x)| \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} [|z(s) - x(s)| + |z(\lambda s) - x(\lambda s)|] d_q s \end{aligned} \quad (6)$$

For $u \in \mathscr{C}(\mathscr{J}, \mathscr{X})$ we consider the operator A: $\mathscr{C}(\mathscr{J}, \mathscr{X}) \to \mathscr{C}(\mathscr{J}, \mathscr{X})$ defined by:

$$(Au)(t) = \epsilon e_\alpha(t^\alpha; q) + bu(t) + \frac{L}{\Gamma_q(\alpha)} \begin{pmatrix} \int_0^t (t-qs)^{(\alpha-1)} u(s) d_q s \\ + \int_0^t (t-qs)^{(\alpha-1)} u(s) d_q s \end{pmatrix}, t \in \mathscr{J}$$

Next, we verify that A is a Picard operator.

For all $t \in \mathscr{J}$, it follows (A2):

$$\begin{aligned} &|(Au)(t) - (A\upsilon)(t)| \\ &\leq b|u(t) - \upsilon(t)| \\ &+ L\left(\frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^t (qs/t; q)_{\alpha-1} [|u(s) - \upsilon(s)| + |u(\lambda s) - \upsilon(\lambda s)|] d_q s \right) \\ &\leq b\|u - \upsilon\|_{C(J,X)} + \frac{2LT^\alpha}{\Gamma_q(\alpha+1)} \|u - \upsilon\|_{\mathscr{C}(\mathscr{J}, \mathscr{X})} \\ &\leq \left(b + \frac{2LT^\alpha}{\Gamma_q(\alpha+1)} \right) \|u - \upsilon\| \mathscr{C}(\mathscr{J}, \mathscr{X}) \\ &= \Omega_{b,L,T,\alpha,q} \|u - \upsilon\| \mathscr{C}(\mathscr{J}, \mathscr{X}) \end{aligned}$$

Thus, A is a contraction via the Chebyshev norm $||\cdot||$ on $\mathscr{C}(\mathscr{J}, \mathscr{X})$ due to (3).

Applying the Banach contraction principle to A, we derive that A is a Picard operator and $F_A = \{u^*\}$. Then for $t \in \mathscr{J}$:

$$u^*(t) = \epsilon e_\upsilon(t^\alpha; q) + bu^*(t) + \frac{L}{\Gamma_q(\alpha)} \begin{pmatrix} \int_0^t (t-qs)^{(\alpha-1)} u^*(s) d_q s \\ + \int_0^t (t-qs)^{(\alpha-1)} u^*(\lambda s) d_q s \end{pmatrix}$$

It remains to verify that the solution u^* is increasing. Indeed, for $0 \leq t_1 < t_2 \leq b$ and denote $m : \min_{s \in J} [u^*(s) + (\lambda s)] \in \mathbb{R}_+$, we have:

$$\begin{aligned} &u^*(t_2) - u^*(t_1) \\ &= \epsilon [e_\alpha(t_2^\alpha; q) - e_\alpha(t_1^\alpha; q)] + b[u^*(t_2) - u*(t_1)] \\ &+ \frac{L}{\Gamma_q(\alpha)} \int_0^{t_1} [(t_1 - qs)^{(\alpha-1)} - (t_2 - qs)^{(\alpha-1)}] \\ &[u^*(s) + u^*(\lambda s)] d_q s \\ &+ \frac{L}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} (t_2 - qs)^{(\alpha-1)} [u^*(s) + u^*(\lambda s)] d_q s \\ &\leq \epsilon [e_\alpha(t_2^\alpha; q) - e_\alpha(t_1^\alpha; q)] + b[u^*(t_2) - u^*(t_1)] \\ &+ \frac{mL}{\Gamma_q(\alpha)} \int_0^{t_1} [t_2^{\alpha-1} (qs/t_2; q)_{\alpha-1} - t_1^{\alpha-1} (qs/t_1; q)_{\alpha-1}] d_q s \\ &+ \frac{mL}{\Gamma_q(\alpha)} \int_{t_1}^{t_2} t_2^{\alpha-1} (qs/t_2; q)_{\alpha-1} d_q s \\ &= \epsilon [e_\alpha(t_2^\alpha; q) - e_\alpha(t_1^\alpha; q)] + b[u^*(t_2) - u^*(t_1)] \\ &+ \frac{mL}{\Gamma_q(\alpha+1)} (t_2^\alpha - t_1^\alpha) > 0 \end{aligned}$$

Then, we obtain u^* is increasing:

$$u^* \leq \epsilon e_\alpha(t^\alpha; q) + bu^* + \frac{2L}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u^*(s) d_q s$$

Using Lemma 4.6 and Remark 4, we get:

$$\begin{aligned} &u^*(t) \leq \frac{\epsilon e_\alpha(t^\alpha; q)}{(1-b)} \\ &+ \frac{1}{(1-b)} \cdot \frac{2L}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} u^*(s) d_q s \\ &u^*(t) \leq C_{e_\alpha} \epsilon e_\alpha(t^\alpha; q) \end{aligned}$$

where, $C_{e_\alpha} := \frac{1}{1-b} \cdot e_\alpha\left(\frac{2L}{1-b} T^\alpha; q \right)$.

In particular, if $u = |z - x|$, from (6), $u \leq Au$ and applying the Lemma 4.5 we obtain $u \leq u^*$, where A is a picard and increasing operator. As a result, we know:

$$|z(t) - x(t)| \leq C_{e_\alpha} \epsilon e_\alpha (t^\alpha; q)$$

Thus, the Equation (1) is *U-H-q*-Mittag-Leffler stable.

Remark 4.10

Theorem 3.1 and 4.9 can easily be extended to the generalized q-fractional multi-pantograph of the form:

$$\begin{aligned} &{}^c D_q^\alpha x(t) = F\left(t, x(t), x(\lambda_1 t), \ldots, x\right), \quad 0 < q > 1, t \in [0, T] \\ &x(0) + g(x) = x_0 \end{aligned}$$

where, ${}^c D_q^\alpha$ is the Caputo q-fractional derivative, $\alpha \in (0, 1)$.

Now we give an example to illustrate our results.

An Example

Consider the nonlinear q-fractional pantograph problem given by:

$$\begin{cases} {}^c D_q^\alpha x(t) = \dfrac{1}{5} + \dfrac{1}{10} x(t) \dfrac{1}{10} x\left(\dfrac{t}{2}\right), t \in [0,1], \\ x(0) + \sum_{i=1}^m a_i x(t_i) = 0, 0 < t_1 < t_2 < \cdots < t_m < 1 \end{cases} \qquad (7)$$

where, $\alpha \in (0, 1)$, $q \in (0, 1)$, $a_i > 0$, $i = 0,1,2,\ldots,m$ are positive constants with:

$$\sum_{i=1}^m a_i \leq \frac{1}{3}$$

Set:

$$F(t, u, \upsilon) = \frac{1}{5} + \frac{1}{10} u + \frac{1}{10} \upsilon, t \in [0,1], u, \upsilon \in \mathscr{X}$$

and:

$$g(x) = \sum_{i=1}^m a_i x(t_i).$$

Let $u, \upsilon, \bar{u}, \bar{\upsilon} \in \mathscr{X}$ and $t \in [0, 1]$. Then we have:

$$\left|F(t, u, \upsilon) - F(t, \bar{u}, \bar{\upsilon})\right| \leq \frac{1}{10}\left(|u - \bar{u}| + |\upsilon - \bar{\upsilon}|\right)$$

On the other hand, we have:

$$\begin{aligned} |g(u) - g(\bar{u})| &\leq \left|\sum_{i=1}^m a_i u - \sum_{i=1}^m a_i \bar{u}\right| \\ &\leq \sum_{i=1}^m a_i |u - \bar{u}| \\ &\leq \frac{1}{3}|u(t) - \bar{u}(t)| \end{aligned}$$

Denote: $\alpha = \frac{1}{8}, L = \frac{1}{10}, q = \frac{1}{2}$ and $b = \frac{1}{3}$.

Thus:

$$L \leq \frac{\Gamma_q(\alpha+1)}{2} \Leftrightarrow \Gamma_q(\alpha+1) \geq 2L = 0.2$$

where, $\frac{\Gamma_q(\alpha+1)}{2} = \frac{0.957935}{2} = 0.4789675$.

Equation (7) follows the inequality:

$$\left|{}^c D_q^\alpha z(t) - F(t, x(t), x(\lambda t))\right| \leq \epsilon e_\alpha \left(t^{\frac{1}{8}}; \frac{1}{2}\right)$$

Now all assumptions in Theorem 3.1 and 4.9 are satisfied, the problem (7) has a unique solution and the Equation (7) is *U-H-q*-Mittag-Leffler stable with:

$$|z(t) - x(t)| \leq C_{e_\alpha} \epsilon e_\alpha \left(t^{\frac{1}{8}}; \frac{1}{2}\right)$$

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Ethics

The Authors declare there is not conflict of interest.

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