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Risks Ratios of Shrinkage Estimators for the Multivariate Normal Mean

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Abstract: We study the estimation of the mean θ of a multivariate Gaussian random variable $X \sim N_p(\theta, \sigma^2 I_p)$ in $\Re^p$, σ^2 is unknown and estimated by the chi-square variable $S^2 \sim \sigma^2 \chi_n^2$. In this work we are interested in studying bounds and limits of risk ratios of shrinkage estimators to the maximum likelihood estimator X, when n and p tend to infinity. We recall that the risk ratios of shrinkage estimators to the maximum likelihood estimator has a lower bound B_m, when n and p tend to infinity. We show that if the shrinkage function $\psi(S^2, \|X^2\|)$ satisfies some conditions, the risk ratios of shrinkage estimators $(1-\psi(S^2, \|X^2\|)S^2/\|X^2\|)X$, which did not inevitably minimax, to attain the limiting lower bound B_m which is strictly lower than 1.

Keywords: James-Stein Estimator, Non-Central Chi-Square Distribution, Quadratic Risk, Shrinkage Estimator

Introduction

The estimation by shrinkage estimators of the mean θ of a multivariate normal distribution $N_p(\theta, \sigma^2 I_p)$ in $\Re^p$ has experienced many developments since the papers of (Stein 1956; James and Stein 1961; Stein 1981). In these works one estimates the mean θ by shrinkage estimators deduced from the empirical mean estimator, which are better in quadratic loss than the empirical mean estimator.

More precisely, if X represents an observation or a sample of a multivariate normal distribution $N_p(\theta, \sigma^2 I_p)$, the aim is to estimate θ by an estimator δ relatively at the quadratic loss function:

$$L(\delta, \theta) = \|\delta - \theta\|_p^2$$

where $\|.\|_p$ is the usual norm in $\Re^p$. To this loss function we associate the risk function:

$$R(\delta, \theta) = E_\theta(L(\delta, \theta)).$$

James and Stein (1961) introduced a class of James-Stein estimators improving the maximum likelihood estimator $\delta_0 = X$, when the dimension of the space parameters $p \geq 3$, noted:

$$\delta_j^{JS} = \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right) X_j; j = 1, \ldots, p \tag{1.1}$$

where $S^2 \sim \sigma^2 \chi_n^2$ is the estimate of σ^2.

Baranchik (1964) proposed the positive-part of James-Stein estimator dominating the James-Stein estimator when $p \geq 3$, noted:

$$\delta_j^{JS+} = \max\left(0, \left(1 - \frac{(p-2)S^2}{(n+2)\|X\|^2}\right)\right) X; j = 1, \ldots, p \tag{1.2}$$

Casella and Hwang (1982) studied the case where σ^2 is known $(\sigma^2 = 1)$ and showed that if $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p} = c (> 0)$, then:

$$\lim_{p\to+\infty}\frac{R(\delta_{JS},\theta)}{R(X,\theta)}=\lim_{p\to+\infty}\frac{R(\delta_{JS}^{+},\theta)}{R(X,\theta)}=\frac{c}{1+c}.$$

Thus, they showed the stability of the dominating of James-Stein estimator and its positive-part, to the maximum likelihood estimator, when the dimension of space parameter p tends to infinity, in the case where σ^2 is known.

Li (1995) has considered the following model:

$$(y_{ij}/\theta_j,\sigma^2)\sim N(\theta_j,\sigma^2),\ i=1,\dots,n,\ j=1,\dots,m$$

where, $E(y_{ij})=\theta_j$ for the group j and $\mathrm{var}(y_{ij})=\sigma^2$ is unknown. He studied the shrinkage estimators $\delta=(\delta_1,\dots,\delta_m)^t$ where:

$$\delta_j=\left(1-\varphi(S^2,T^2)\frac{S^2}{T^2}\right)(\bar{y}_j-\bar{y})+\bar{y}$$

with:

$$S^2=\sum_{i=1}^{n}\sum_{j=1}^{m}(y_{ij}-\bar{y}_j)^2,$$
$$T^2=n\sum_{j=1}^{m}(\bar{y}_j-\bar{y})^2,$$

and:

$$\bar{y}_j=\frac{\sum_{i=1}^{n}y_{ij}}{n},\ \bar{y}=\frac{\sum_{j=1}^{m}\bar{y}_j}{m}.$$

The James-Stein estimators are written in this case:

$$\delta^{JS}=(\delta_1^{JS},\dots,\delta_m^{JS})^t$$

where:

$$\delta_j^{JS}=\left(1-\frac{(m-3)S^2}{(N+2)T^2}\right)(\bar{y}_j-\bar{y})+\bar{y},\ j=1,\dots,m$$

with $N=(n-1)m$.

In this case, it is clear that the maximum likelihood estimator is $\delta^0=\bar{y}_j$.

Li (1995) has given a lower bound for the ratio $\frac{R(\delta,\theta)}{R(\delta^0,\theta)}$, which allows him to conclude that:

$$\lim_{m\to+\infty}\frac{R(\delta^{JS},\theta)}{R(\delta^0,\theta)}=\lim_{m\to+\infty}\frac{R(\delta^{JS+},\theta)}{R(\delta^0,\theta)}=\frac{q}{q+\frac{\sigma^2}{n}}$$

provided that $\lim_{m\to+\infty}\sum_{j=1}^{m}(\theta_j-\bar{\theta})^2/m=q$ exists.

Benmansour and Hamdaoui (2011) interested the case where σ^2 is unknown. The authors showed that if $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, then the risk ratio of James-Stein estimator δ^{JS} to the maximum likelihood estimator X, tends to $\frac{\frac{2}{n+2}+c}{1+c}$ when p tends to infinity and n is fixed. Under the same condition namely $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, they showed that the risk ratio of James-Stein estimator δ^{JS} to the maximum likelihood estimator X, tends to the value $\frac{c}{1+c}$ when n and p tend simultaneously to infinity. They also found the same results for the positive-part of James-Stein estimator δ^{JS+}.

Hamdaoui and Benmansour (2015) studied the behavior of risk ratios of the general class of shrinkage estimator proposed by Benmansour and Mourid (2007), given by $\delta_{l,\delta^{JS},\psi}=\delta_{l,\delta^{JS}}=\delta^{JS}+l\psi(S^2,\|X^2\|)X$, in the case where σ^2 is unknown. Then, they showed that if $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$, the risk ratio of shrinkage estimator $\delta_{l,\delta^{JS},\psi}$, tends to a value less than 1, when n and p tend simultaneously to infinity and provided that the function ψ satisfies certain conditions.

In this study we adopt the same model $X\sim N_p(\theta,\sigma^2I_p)$ and independently of the observation X, we observe $S^2\sim\sigma^2\chi_n^2$ an estimator of σ^2. Note that $R(X,\theta)=p\sigma^2$ is the risk of the maximum likelihood estimator. We generalize the results given by Casella and Hwang (1982), Benmansour and Hamdaoui (2011) and Hamdaoui and Benmansour (2015), by studying the class of shrinkage estimators $\delta=\left(1-\psi(S^2,\|X^2\|)\frac{S^2}{\|X^2\|}\right)X$, which is containing the estimators δ^{JS} and δ^{JS+}. Then we show that if $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c(>0)$ and the shrinkage function ψ satisfies some conditions different from the ones given in Hamdaoui and Benmansour (2015), the risk ratio of the estimator δ to the maximum likelihood estimator X, tends to the value $\frac{c}{1+c}$ when n and p tend simultaneously to infinity.

In the following we denote the general form of shrinkage estimator as follows:

$$\delta = \left(1 - \varphi\left(S^2, \| X^2 \|\right)\right) X \tag{1.3}$$

In Section 1, we recall some results obtained in Hamdaoui and Benmansour (2015). The authors showed, that under the condition $\lim_{p \to +\infty} \frac{\| \theta \|^2}{p\sigma^2} = c(>0)$, the risk ratio of the shrinkage estimator δ given in (1.3), to the maximum likelihood estimator *X*, has a lower bound $B_m = \frac{c}{1+c}$, when *n* and *p* tend to infinity. The second result indicates that under the same condition $\lim_{p \to +\infty} \frac{\| \theta \|^2}{p\sigma^2} = c(>0)$, the risk ratio of James-Stein estimator δ^{JS} given in (1.1), to the maximum likelihood estimator *X*, tends to the value $\frac{c}{1+c}$ when *n* and *p* tend simultaneously to infinity.

In Section 2, we give the main results of this paper. We considered the general class of shrinkage estimators $\delta = \left(1 - \psi\left(S^2, \| X^2 \|\right) \frac{S^2}{\| X^2 \|}\right) X$, which did not inevitably minimax and we show that, if the shrinkage function ψ satisfies certain conditions which is different from the ones given in Hamdaoui and Benmansour (2015), the risk ratio of δ to the maximum likelihood estimator, to attain the limiting lower bound B_m provided that $\lim_{p \to +\infty} \frac{\| \theta \|^2}{p\sigma^2} = c$.

In the end we graph the corresponding risks ratios for the estimators: James-Stein δ^{JS}, its positive-part δ^{JS+} and estimators defined in selected examples for divers values of *n* and *p*.

Preliminaries

We recall that if *X* is a multivariate Gaussian random $N_p\left(\theta, \sigma^2 I_p\right)$ in $\Re^p$ then $\frac{\|X\|^2}{\sigma^2} \sim \chi_p^2(\lambda)$ where $\chi_p^2(\lambda)$ denotes the non-central chi-square distribution with *p* degrees of freedom and non-centrality parameter $\lambda = \frac{\| \theta \|^2}{2\sigma^2}$.

We recall the following lemma given in Fourdrinier et al. (2008), that we will use often in our proofs.

Lemma 2.1

Let $X \sim N_p\left(\theta, \sigma^2 I_p\right)$ with $\theta \in \Re^p$. Then, for any $p \geq 3$, we have:

$$E\left(\frac{1}{\| X^2 \|}\right) = \frac{1}{\sigma^2} E\left(\frac{1}{p-2+2K}\right) \tag{2.1}$$

And for any $p \geq 5$, we have:

$$E\left(\frac{1}{\left(\| X^2 \|\right)}\right) = \frac{1}{\sigma^4} E\left(\frac{1}{(p-2+2K)(p-4+2K)}\right) \tag{2.2}$$

where, $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$ being the Poisson's distribution of parameter $\frac{\| \theta \|^2}{2\sigma^2}$.

Theorem 2.2 (Hamdaoui and Benmansour, 2015)

The risk of estimator given in (1.3) is:

$$R(\delta, \theta) = \sigma^2 E\left\{\varphi_K^2 \chi_{p+2K}^2 - 2\varphi_K\left(\chi_{p+2K}^2 - 2K\right) + p\right\}$$

where, $\varphi_K = \varphi\left(\sigma^2 \chi_n^2, \sigma^2 \chi_{p+2K}^2\right)$ and $K \sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right)$.

Furthermore:

$$R(\delta, \theta) \geq B_p(\theta)$$

with:

$$B_p(\theta) = \sigma^2 E\left\{p - 2 - E\left(\frac{(p-2)}{p-2+2K}\right)\right\}.$$

Note that $P\left(\frac{\| \theta \|^2}{2\sigma^2}\right)$ being the Poisson's distribution of parameter $\frac{\| \theta \|^2}{2\sigma^2}$.

We set: $b_p(\theta) = \frac{B_p(\theta)}{R(X, \theta)}$. It is clear that if $\lim_{p \to +\infty} \frac{\| \theta \|^2}{p\sigma^2} = c$, then:

$$\lim_{p \to +\infty} b_p(\theta) = \frac{c}{1+c} = B_m \tag{2.3}$$

In the particular case where $\varphi\left(S^2, \| X^2 \|\right) = d\frac{S^2}{\| X^2 \|}$, we have $\delta_d = \left(1 - d\frac{S^2}{\| X^2 \|}\right) X$, hence:

$$R(\delta_d, \theta) = \sigma^2 E\left\{p + n\left[d^2(n+2) - 2d(p-2)\right] E\left(\frac{1}{p-2+2K}\right)\right\}.$$

For $d = \frac{p-2}{n+2}$, we obtain the James-Stein estimator (defined in (1.1)) which minimizes the risk of δ_d; whose quadratic risk is:

$$R\left(\delta^{JS},\theta\right) = \sigma^2 E\left\{p - \frac{n}{n+2}(p-2)^2 E\left(\frac{1}{p-2+2K}\right)\right\}.$$

Proposition 2.3 (Hamdaoui and Benmansour (2015))

Let δ is given in (1.3), if $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, then:

$$\lim_{n,p\to+\infty} \frac{R(\delta,\theta)}{R(X,\theta)} \geq \frac{c}{1+c} \tag{2.4}$$

$$\lim_{n,p\to+\infty} \frac{R\left(\delta^{JS},\theta\right)}{R(X,\theta)} = \frac{c}{1+c} \tag{2.5}$$

Main Results

Limit of Risk Ratios of Shrinkage Estimators

We now rewrite the estimator in (1.3) by letting:

$$\varphi\left(S^2,\|X^2\|\right) = \psi\left(S^2,\|X^2\|\right)\frac{S^2}{\|X^2\|}$$

is given by:

$$\delta_j = \left(1-\psi\left(S^2,\|X^2\|\right)\frac{S^2}{\|X^2\|}\right)X_j,\ j=1,\ldots.p \tag{3.1}$$

Theorem 3.1

Assume that δ_j is given in (3.1), such that $p \geq 5$ and ψ satisfies:

- $\frac{\psi\left(S^2,\|X^2\|\right)}{p-2}$ converge in probability to $\frac{1}{n+2}$ when $p \to +\infty$.
- $\left|\frac{\psi\left(S^2,\|X^2\|\right)}{p-2}\right| \leq g\left(S^2\right)$ a.s; where:

$$E\left[\left(g^2\left(S^2\right)\right)^{1+y}\right] = O\left(\frac{1}{n^{2(1+y)}}\right)\ \textit{for some}\ \gamma > 0$$

If $\lim_{p\to+\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$, then:

$$\lim_{n,p\to+\infty} \frac{R(\delta,\theta)}{R(X,\theta)} = \frac{c}{1+c}.$$

Proof:

$$\begin{aligned} R(\delta,\theta) &= E\left\{\sum_{J=1}^{p}\left(\delta_J - \theta_J\right)^2\right\} \\ &= E\left\{\sum_{J=1}^{p}\left(\delta_J - \delta_j^{JS} + \delta_j^{JS} - \theta_J\right)^2\right\} \\ &= R\left(\delta^{JS},\theta\right) + E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)^2\right\} \\ &\quad + 2E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)\left(\delta_j^{JS} - \theta_j\right)\right\}. \end{aligned}$$

We write:

$$\Delta_{JS} = R(\delta,\theta) - R\left(\delta^{JS},\theta\right)$$

then:

$$\begin{aligned} \Delta_{JS} &= E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)^2\right\} \\ &\quad + 2E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)\left(\delta_j^{JS} - \theta_j\right)\right\}, \end{aligned}$$

thus:

$$\frac{\Delta_{JS}}{p\sigma^2} = \Delta_1 + \Delta_2$$

where:

$$\Delta_1 = \frac{1}{p\sigma^2} E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)^2\right\} \tag{3.2}$$

and:

$$\Delta_2 = \frac{2}{p\sigma^2} E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)\left(\delta_j^{JS} - \theta_j\right)\right\} \tag{3.3}$$

$$\begin{aligned} \bullet\Delta_1 &= \frac{1}{p\sigma^2} E\left\{\sum_{J=1}^{p}\left(\delta_j - \delta_j^{JS}\right)^2\right\} \\ &= \frac{1}{p\sigma^2} E\left\{\sum_{J=1}^{p}\left(\frac{p-2}{n+2} - \psi\left(S^2,\|X\|^2\right)\right)^2\left(\frac{S^2}{\|X\|^2}\right)^2 X_J^2\right\} \\ &= \frac{(p-2)^2}{p\sigma^2} E\left\{\left(\frac{1}{n+2} - \frac{\psi\left(S^2,\|X\|^2\right)}{p-2}\right)^2 \frac{\left(S^2\right)^2}{\|X\|^2}\right\} \end{aligned} \tag{3.4}$$

We write:

$$\psi_p = \frac{\psi\left(S^2,\|X\|^2\right)}{p-2}$$

and let $\varepsilon > 0$. Then we have:

$$\bullet\Delta_1 = \frac{(p-2)^2}{p\sigma^2} \times E\left\{ \begin{array}{l} \left(\frac{1}{n+2}-\psi_p\right)^2 \frac{(S^2)^2}{\|X\|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 \le \varepsilon\right)} \\ +\left(\frac{1}{n+2}-\psi_p\right)^2 \frac{(S^2)^2}{\|X\|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)} \end{array} \right\}$$

$$\le \frac{(p-2)^2}{p\sigma^2}\varepsilon E\left\{\frac{(S^2)^2}{\|X\|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 \le \varepsilon\right)}\right\} + \frac{(p-2)^2}{p\sigma^2} E\left\{\left(\frac{1}{n+2}-\psi_p\right)^2 \frac{(S^2)^2}{\|X\|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\}.$$

We set:

$$\alpha_1(n,p) = \frac{(p-2)^2}{p\sigma^2}\varepsilon E\left\{\frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}\psi_p\right|^2 \le \varepsilon\right)}\right\}$$

and:

$$\alpha_2(n,p) = \frac{(p-2)^2}{p\sigma^2}\varepsilon E\left\{\left(\frac{1}{n+2}-\psi_p\right)^2 \frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}\psi_p\right|^2 > \varepsilon\right)}\right\}.$$

Using Schwarz's inequality, we have:

$$\bullet\alpha_1(n,p) \le \frac{(p-2)^2}{p\sigma^2}\varepsilon E^{1/2}\left\{\left(\frac{(S^2)^2}{\| X \|^2}\right)\right\} P^{1/2}\left(\left|\frac{1}{n+2}-\psi_p\right|^2 \le \varepsilon\right).$$

From the independence of $\|X\|^2$ and S^2 and that $E\left(\frac{1}{(\chi_P(\lambda))^2}\right) \le \frac{1}{(p-2)(p-4)}$ (See formula 2.2 of Lemma 2.1), we deduce that:

$$\alpha_1(n,p) \le \frac{(p-2)^2}{p}\varepsilon\sqrt{\frac{n(n+2)(n+4)(n+6)}{(p-2)(p-4)}} \times P^{1/2}\left(\left|\frac{1}{n+2}-\psi_p\right|^2 \le \varepsilon\right) \le \frac{(p-2)^2}{p}\varepsilon\sqrt{\frac{n(n+2)(n+4)(n+6)}{(p-2)(p-4)}}.$$

For ε sufficiently small, it is clear that $\alpha_1(n,p) = 0$, hence $\underline{\lim}_{n,p\to+\infty}\alpha_1(n,p) \le 0$.

Now, we show that $\underline{\lim}_{n,p\to+\infty}\alpha_2(n,p) \le 0$, indeed:

$$\bullet\alpha_2(n,p) = \frac{(p-2)^2}{p\sigma^2} \times E\left\{\left(\frac{1}{n+2}-\psi_p\right)^2 \frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\}$$
$$\le \frac{2(p-2)^2}{p\sigma^2} \times E\left\{\left(\frac{1}{(n+2)^2}+g^2(S^2)\right)\frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\} \tag{3.5}$$
$$\le \frac{2(p-2)^2}{p\sigma^2}\frac{1}{(n+2)^2} E\left\{\frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\} + \frac{2(p-2)^2}{p\sigma^2} E\left\{g^2(S^2)\frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\}$$

The inequality (3.5) according to the second condition and the following inequality: for any $a,b \in \Re$, $-2ab \le a^2+b^2$.

We set:

$$\alpha_{21}(n,p) = \frac{(p-2)^2}{p\sigma^2}\frac{1}{(n+2)^2} E\left\{\frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\}$$

and:

$$\alpha_{22}(n,p) = \frac{(p-2)^2}{p\sigma^2} E\left\{g^2(S^2)\frac{(S^2)^2}{\| X \|^2} I_{\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)}\right\}.$$

From Schwarz's inequality, we have:

$$\bullet\alpha_{21}(n,p) \le \frac{(p-2)^2}{p\sigma^2}\frac{1}{(n+2)^2} \times E^{1/2}\left(\left\{\frac{(S^2)^4}{\|X\|^4}\right\}\right) P^{1/2}\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right)$$
$$\le \frac{(p-2)^2}{p}\frac{1}{(n+2)^2} \times \sqrt{\frac{n(n+2)(n+4)(n+6)}{(p-2)(p-4)}} P^{1/2}\left(\left|\frac{1}{n+2}-\psi_p\right|^2 > \varepsilon\right).$$

Thus, it is clear that $\underline{\lim}_{n,p\to+\infty}\alpha_{21}(n,p) \le 0$, because ψ_p converge in probability to $\frac{1}{n+2}$.

From Holder's inequality, we have:

$$\bullet\, \alpha_{22}(n,p) \leq \frac{(p-2)^2}{p\sigma^2}$$

$$\times E^{\frac{1+\gamma/2}{1+\gamma}}\left(\left\{g^2(S^2)\frac{(S^2)^2}{\|X\|^2}\right\}^{\frac{1+\gamma}{1+\gamma/2}}\right)P^{\frac{\gamma/2}{1+\gamma}}\left(\left|\frac{1}{n+2}-\psi_p\right|^2>\varepsilon\right)$$

$$\leq \frac{(p-2)^2}{p\sigma^2}E^{\frac{1+\gamma/2}{1+\gamma}\frac{1}{1+\gamma/2}}\left(\left\{g^2(S^2)\right\}^{\frac{1+\gamma}{1+\gamma/2}(1+\gamma/2)}\right)$$

$$\times E^{\frac{1+\gamma/2}{1+\gamma}\frac{\gamma/2}{1+\gamma/2}}\left(\left\{\frac{(S^2)^2}{\|X\|^2}\right\}^{\frac{1+\gamma}{1+\gamma/2}\frac{1+\gamma/2}{\gamma/2}}\right)P^{\frac{\gamma/2}{1+\gamma}}\left(\left|\frac{1}{n+2}-\psi_p\right|^2>\varepsilon\right)$$

$$\leq \frac{(p-2)^2}{p\sigma^2}E^{\frac{1}{1+\gamma}}\left(\left\{g^2(S^2)\right\}^{1+\gamma}\right)$$

$$\times E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{\frac{(S^2)^2}{\|X\|^2}\right\}^{\frac{1+\gamma}{\gamma/2}}\right)P^{\frac{\gamma/2}{1+\gamma}}\left(\left|\frac{1}{n+2}-\psi_p\right|^2>\varepsilon\right)$$

$$\leq \frac{(p-2)^2}{p\sigma^2}E^{\frac{1}{1+\gamma}}\left(\left\{g^2(S^2)\right\}^{1+\gamma}\right)E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{(S^2)^2\right\}^{\frac{1+\gamma}{\gamma/2}}\right)$$

$$\times E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{\frac{1}{\|X\|^2}\right\}^{\frac{1+\gamma}{\gamma/2}}\right)P^{\frac{\gamma/2}{1+\gamma}}\left(\left|\frac{1}{n+2}-\psi_p\right|^2>\varepsilon\right).$$

The last inequality follows from the independence of $\|X\|^2$ and S^2.

As:

$$E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{(S^2)^2\right\}^{\frac{1+\gamma}{\gamma/2}}\right)=E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{(\sigma^2\chi_n^2)^2\right\}^{\frac{1+\gamma}{\gamma/2}}\right)$$

and:

$$E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{\frac{1}{\|X\|^2}\right\}^{\frac{1+\gamma}{\gamma/2}}\right)=E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{\frac{1}{\sigma^2\chi_{p+2K}}\right\}^{\frac{1+\gamma}{\gamma/2}}\right)$$

From Stirling's formula which expresses that in the neighborhood of $+\infty$, we have:

$$\Gamma(y+1)\cong\sqrt{2\pi}\,y^{y+\frac{1}{2}}e^{-y}$$

and the fact that:

$$\lim_{n\to+\infty}\left(1+\frac{y}{n}\right)=e^{y}$$

we obtain:

$$E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{(S^2)^2\right\}^{\frac{1+\gamma}{\gamma/2}}\right)=4(\sigma^2)^2\left[\frac{\Gamma\left(\frac{n}{2}+\frac{4}{\gamma}+4\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^{\frac{\gamma}{2(1+\gamma)}}$$

$$\cong 4(\sigma^2)^2\left(\frac{n}{2}+\frac{4}{\gamma}+3\right)^2$$

in the neighborhood of $+\infty$ and:

$$E^{\frac{\gamma/2}{1+\gamma}}\left(\left\{\frac{1}{\|X\|^2}\right\}^{\frac{1+\gamma}{\gamma/2}}\right)=\frac{1}{2\sigma^2}E^{\frac{\gamma/2}{1+\gamma}}\left[\frac{\Gamma\left(\frac{p+2K}{2}-\frac{1+\gamma}{\gamma/2}\right)}{\Gamma\left(\frac{p+2K}{2}\right)}\right]$$

$$\cong\frac{1}{\sigma^2}E^{\frac{\gamma/2}{1+\gamma}}\left[\left(\frac{1}{p-6-\frac{4}{\gamma}+2K}\right)^{\frac{1+\gamma}{\gamma/2}}\right]$$

in the neighborhood of $+\infty$.

For p sufficiently large, we have:

$$E\left(\frac{1}{p-6-\frac{4}{\gamma}+2K}\right)^{\frac{1+\gamma}{\gamma/2}}\leq\left(\frac{1}{p-6-\frac{4}{\gamma}}\right)^{\frac{1+\gamma}{\gamma/2}}.$$

Then:

$$E^{\frac{\gamma/2}{1+\gamma}}\left[\left(\frac{1}{p-6-\frac{4}{\gamma}+2K}\right)^{\frac{1+\gamma}{\gamma/2}}\right]\leq\frac{1}{p-6-\frac{4}{\gamma}}.$$

Thus:

$$\alpha_{22}(n,p)\leq\frac{(p-2)^2}{p}\left[4\left(\frac{n}{2}+\frac{4}{\gamma}+3\right)^2\right]$$

$$\times\left(\frac{1}{p-6-\frac{4}{\gamma}}\right)E^{\frac{1}{1+\gamma}}\left(\left\{g^2(S^2)\right\}^{1+\gamma}\right)P^{\frac{\gamma/2}{1+\gamma}}\left(\left|\frac{1}{n+2}-\psi_p\right|^2>\varepsilon\right).$$

As $E\left[\left(g^2\left(S^2\right)^{1+\gamma}\right)\right]=O\left(\frac{1}{n^{2(1+\gamma)}}\right)$ and the fact that $\psi_p \xrightarrow{p\to+\infty} \frac{1}{n+2}$ in probability, it is clear that $\lim_{n,p\to+\infty}\alpha_{22}(n,p)\leq 0$. Hence $\Delta_1 \xrightarrow{n,p\to+\infty} 0$.

From (3.3) and by using Schwarz's inequality we have:

$$\begin{aligned}|\Delta_2| &\leq \frac{2}{p\sigma^2}E\left[\left\{\sum_{j=1}^{p}\left(\delta_j-\delta_j^{JS}\right)^2\right\}^{1/2}\left\{\sum_{j=1}^{p}\left(\delta_j^{JS}-\theta_j\right)^2\right\}^{1/2}\right]\\ &\leq \frac{2}{p\sigma^2}E^{1/2}\left\{\sum_{j=1}^{p}\left(\delta_j-\delta_j^{JS}\right)^2\right\}E^{1/2}\left\{\sum_{j=1}^{p}\left(\delta_j^{JS}-\theta_j\right)^2\right\}\\ &\leq 2\left[\Delta_1\frac{R\left(\delta^{JS},\theta\right)}{p\sigma^2}\right]^{1/2}.\end{aligned}$$

Then $\Delta_2 \xrightarrow{n,p\to+\infty} 0$. Thus, from formula (2.5) of Proposition 2.3, we have:

$$\lim_{n,p\to+\infty}\frac{R(\delta,\theta)}{R(X,\theta)}\leq\frac{c}{1+c}.$$

Hence by using the formula (2.4) of Proposition 2.3, we obtain:

$$\lim_{n,p\to+\infty}\frac{R(\delta,\theta)}{R(X,\theta)}=\frac{c}{1+c}.$$

Example 3.2

Assume the estimator given in (3.1), such that:

$$\psi_1\left(S^2,\|X\|^2\right)=\frac{p-2}{n+2}\frac{\|X\|^2}{\|X\|^2+1}$$
$$i.e.\ \delta_{\psi_1}\left(S^2,\|X\|^2\right)=\left(1-\frac{p-2}{n+2}\frac{S^2}{\|X\|^2+1}\right)X.$$

To show that the function $\psi_1\left(S^2,\|X\|^2\right)$, satisfies the conditions of Theorem 3.1, we used the following lemma.

Lemma 3.3

For any $a > 0$, we have:

$$\frac{1}{m+2+a}\leq E\left(\frac{1}{\chi^2_{m+2}+a}\right)\leq\frac{1}{m+a}..$$

Proof

From Jensen inequality, we have:

$$E\left(\frac{1}{\chi^2_{m+2}+a}\right)\geq\frac{1}{m+2+a}.$$

As:

$$1=E\left(\frac{\chi^2_m}{\chi^2_m+a}\right)+aE\left(\frac{1}{\chi^2_m+a}\right),$$

and using the formula (1.2) in Benmansour and Hamdaoui (2011), we have:

$$1=mE\left(\frac{1}{\chi^2_{m+2}+a}\right)+aE\left(\frac{1}{\chi^2_m+a}\right)$$

then, from Jensen inequality, we obtain:

$$E\left(\frac{1}{\chi^2_{m+2}+a}\right)=\frac{1}{m}\left\{1-aE\left(\frac{1}{\chi^2_m+a}\right)\right\}\leq\frac{1}{m+a}.$$

Now, we show that $\psi_1\left(S^2,\|X\|^2\right)$ satisfies conditions of Theorem 3.1.

Indeed:

$$\begin{aligned}\bullet\ E\left(\frac{\|X\|^2}{\|X\|^2+1}\right)&=E\left(\frac{\chi^2_{p+2K}}{\chi^2_{p+2K}+\frac{1}{\sigma^2}}\right)\\ &=E\left\{E\left(\frac{\chi^2_{p+2K}}{\chi^2_{p+2K}+\frac{1}{\sigma^2}}/K\right)\right\}\\ &=E\left\{(p+2K)E\left(\frac{1}{\chi^2_{p+2+2K}+\frac{1}{\sigma^2}}/K\right)\right\}.\end{aligned}$$

The above equality according of formula (1.2) in Benmansour and Hamdaoui (2011). From lemma 3.3, we have:

$$\begin{aligned}&E\left(\frac{(p+2K)}{p+2+2K+\frac{1}{\sigma^2}}\right)\leq E\left\{(p+2K)E\left(\frac{1}{\chi^2_{p+2+2K}+\frac{1}{\sigma^2}}/K\right)\right\}\\ &\leq E\left(\frac{(p+2K)}{p+2K+\frac{1}{\sigma^2}}\right).\end{aligned}$$

On the one hand, we have:

$$E\left(\frac{(p+2K)}{p+2K+\frac{1}{\sigma^2}}\right)\leq\left(p+\frac{\|\theta\|^2}{\sigma^2}\right)E\left(\frac{1}{p+2K+\frac{1}{\sigma^2}}\right)$$

because the covariance of two functions one increasing and the other decreasing is non-positive, with:

$$K\sim P\left(\frac{\|\theta\|^2}{2\sigma^2}\right).$$

By using lemma 3.1 of Li (1995), we have:

$$E\left(\frac{(p+2K)}{p+2K+\frac{1}{\sigma^2}}\right)\leq\frac{p+\frac{\|\theta\|^2}{\sigma^2}}{p-2+\frac{\|\theta\|^2}{\sigma^2}+\frac{1}{\sigma^2}}\xrightarrow{p\to+\infty}\frac{1+c}{1+c}=1.$$

On the other hand, we have:

$$E\left(\frac{(p+2K)}{p+2+2K+\frac{1}{\sigma^2}}\right)=1-\left(2+\frac{1}{\sigma^2}\right)E\left(\frac{1}{p+2+2K+\frac{1}{\sigma^2}}\right).$$

Using lemma 3.1 of Li (1995), we have:

$$E\left(\frac{(p+2K)}{p+2K+\frac{1}{\sigma^2}}\right)\geq\frac{p-2+\frac{\|\theta\|^2}{\sigma^2}}{p+\frac{\|\theta\|^2}{\sigma^2}+\frac{1}{\sigma^2}}\xrightarrow{p\to+\infty}\frac{1+c}{1+c}=1.$$

Thus:

$$\lim_{p\to+\infty}E\left(\frac{\|X\|^2}{\|X\|^2+1}\right)=1.$$

Let $a>0$ and using Markov's inequality, we have:

$$P\left(\left|\frac{\|X\|^2}{\|X\|^2+1}-1\right|>a\right)\leq\frac{E\left(\left|\frac{\|X\|^2}{\|X\|^2+1}-1\right|\right)}{a}$$
$$=\frac{E\left(1-\frac{\|X\|^2}{\|X\|^2+1}\right)}{a}\xrightarrow{p\to+\infty}0.$$

Therefore, the function ψ_1 satisfies the first condition of Theorem 3.1.

For the second condition it suffices to take

$$g(S^2)=\frac{1}{n+2}.$$

Remark 3.4

- It is obvious that the James-Stein estimator δ^{JS} satisfies the conditions of Theorem 3.1, so we give another proof that the James-Stein estimator δ^{JS} dominating the maximum likelihood estimator *X*, even if the dimension of parameter space p and the sample size *n* tend to infinity
- We also note that any shrinkage estimator dominating the James-Stein estimator dominates the maximum likelihood estimator even if the dimension of parameter space *p* and the sample size *n* tend to infinity

The following Proposition gives the same results of Theorem 3.1 with different conditions on ψ.

Proposition 3.5

Assume that δ_j is given in (3.1) and that ψ satisfies:

$$\left|\frac{1}{n+2}-\frac{\psi(S^2,\|X^2\|)}{p-2}\right|\leq g(S^2)a.s$$

where the function *g* is monotone non-increasing such that:

$$E\left[\left(g(S^2)\right)^2\right]=O\left(\frac{1}{n^{2+\gamma}}\right)\text{ where }\gamma>0$$

If $\lim_{p\to+\infty}\frac{\|\theta\|^2}{p\sigma^2}=c$, then:

$$\lim_{n,p\to+\infty}\frac{R(\delta,\theta)}{R(X,\theta)}=\frac{c}{1+c}.$$

Proof:

$$R(\delta,\theta)=E\left\{\sum_{J=1}^{p}(\delta_J-\theta_J)^2\right\}$$
$$=E\left\{\sum_{J=1}^{p}\left(\delta_J-\delta_j^{JS}+\delta_j^{JS}-\theta_J\right)^2\right\}$$
$$=R(\delta^{JS},\theta)+E\left\{\sum_{J=1}^{p}\left(\delta_j-\delta_j^{JS}\right)^2\right\}$$
$$+2E\left\{\sum_{J=1}^{p}\left(\delta_j-\delta_j^{JS}\right)\left(\delta_j^{JS}-\theta_j\right)\right\}.$$

We write:

$$\Delta_{JS}=R(\delta,\theta)-R(\delta^{JS},\theta)$$

then:

$$\frac{\Delta_{JS}}{p\sigma^2} = \Delta_1 + \Delta_2$$

where Δ_1 and Δ_2 are given in (3.2) and (3.3).

From formula (3.4) and the independence of $\|X\|^2$ and S^2, we have:

$$\begin{aligned}\Delta_1 &= \frac{(p-2)^2}{p\sigma^2} E\left\{\left(\frac{1}{n+2} - \frac{\psi\left(S^2,\|X^2\|\right)}{p-2}\right)^2 \frac{\left(S^2\right)^2}{\|X^2\|}\right\} \\ &\le \frac{(p-2)^2}{p\sigma^2} E\left(\frac{1}{\|X^2\|}\right) E\left\{\left(g\left(S^2\right)\right)^2 \left(S^2\right)^2\right\} \\ &\le \frac{(p-2)^2}{p\sigma^4} E\left(\frac{1}{p-2+2K}\right) E\left\{\left(g\left(S^2\right)\right)^2\right\} E\left\{\left(S^2\right)^2\right\}.\end{aligned}$$

The last inequality comes from the fact that the covariance of two functions, one increasing and the other decreasing is non-positive.

As $E\left(\frac{1}{p-2+2K}\right) \le \frac{1}{p-2}$ and $E\left[\left(g\left(S^2\right)\right)^2\right] = O\left(\frac{1}{n^{2+\gamma}}\right)$, we have:

$$\begin{aligned}\Delta_1 &\le \frac{(p-2)n(n+2)}{p} E\left\{\left(g\left(S^2\right)\right)^2\right\} \\ &\le \frac{(p-2)n(n+2)}{p} M \frac{1}{n^{2+\gamma}}\end{aligned}$$

where, M is a positive constant.

Thus, it is clear that $\lim_{n,p\to+\infty} \Delta_1 = 0$.

To show that $\lim_{n,p\to+\infty} \Delta_2 = 0$ using the Schwarz's inequality, we have:

$$\begin{aligned}|\Delta_2| &\le \frac{2}{p\sigma^2} E\left[\left\{\sum_{j=1}^{p}\left(\delta_j - \delta_j^{JS}\right)^2\right\}^{1/2} \left\{\sum_{j=1}^{p}\left(\delta_j^{JS} - \theta_j\right)^2\right\}^{1/2}\right] \\ &\le \frac{2}{p\sigma^2} E^{1/2}\left\{\sum_{j=1}^{p}\left(\delta_j - \delta_j^{JS}\right)^2\right\} E^{1/2}\left\{\sum_{j=1}^{p}\left(\delta_j^{JS} - \theta_j\right)^2\right\} \\ &\le 2\left[\Delta_1 \frac{R\left(\delta^{JS},\theta\right)}{p\sigma^2}\right]^{1/2}.\end{aligned}$$

then $\Delta_2 \xrightarrow{n,p\to+\infty} 0$. Thus, from formula (2.5) of Proposition 2.3, we have:

$$\lim_{n,p\to+\infty} \frac{R(\delta,\theta)}{R(X,\theta)} \le \frac{c}{1+c}.$$

Hence by using the formula (2.4) of Proposition 2.3, we obtain:

$$\lim_{n,p\to+\infty} \frac{R(\delta,\theta)}{R(X,\theta)} = \frac{c}{1+c}.$$

Example 3.6

Assume the estimator given in (3.1), such that:

$$\psi_2\left(S^2,\|X\|^2\right) = \frac{p-2}{n+2} + \frac{p-2}{n+2}\frac{1}{S^2}$$

$$\text{i.e. } \delta_{\psi_2}\left(S^2,\|X\|^2\right) = \left(1 - \frac{p-2}{n+2}\frac{\left(S^2+1\right)}{\|X\|^2}\right)X.$$

It is clear that the function ψ_2 satisfies conditions of Proposition 3.5, it suffices to take:

$$g\left(S^2\right) = \frac{1}{n+2}\frac{1}{S^2}.$$

Simulations

We recall the forms of estimators given in Example 3.2, i.e.

$$\psi_1\left(S^2,\|X\|^2\right) = \frac{p-2}{n+2}\frac{\|X\|^2}{\|X\|^2+1},$$

and

$$\delta_{\psi_1}\left(S^2,\|X\|^2\right) = \left(1 - \frac{p-2}{n+2}\frac{S^2}{\|X\|^2+1}\right)X$$

And in the Example 3.6, i.e.:

$$\psi_2\left(S^2,\|X\|^2\right) = \frac{p-2}{n+2} + \frac{p-2}{n+2}\frac{1}{S^2}$$

and

$$\delta_{\psi_2}\left(S^2,\|X\|^2\right) = \left(1 - \frac{p-2}{n+2}\frac{\left(S^2+1\right)}{\|X\|^2}\right)X.$$

of which we illustrate graphically their risks ratios as well as those of James-Stein and the positive-part of James-Stein denoted respectively:

$$\frac{R\left(\delta_{\psi_1},\theta\right)}{R(X,\theta)}, \frac{R\left(\delta_{\psi_2},\theta\right)}{R(X,\theta)}, \frac{R\left(\delta^{JS},\theta\right)}{R(X,\theta)}, \frac{R\left(\delta^{JS+},\theta\right)}{R(X,\theta)}$$

for various values of n and p.

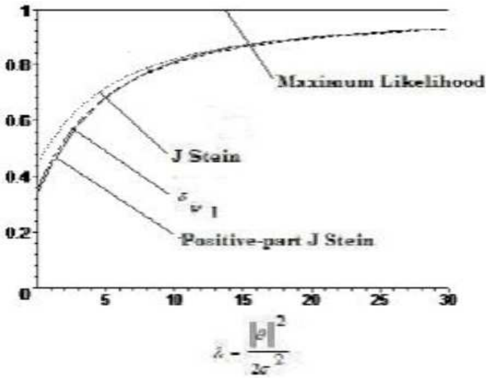

Fig. 1. Graph of risk ratios $\frac{R(\delta_{\psi_1},\theta)}{R(X,\theta)}, \frac{R(\delta^{JS},\theta)}{R(X,\theta)}, \frac{R(\delta^{JS+},\theta)}{R(X,\theta)}$ as function of $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ for $n = 10$ and $p = 6$

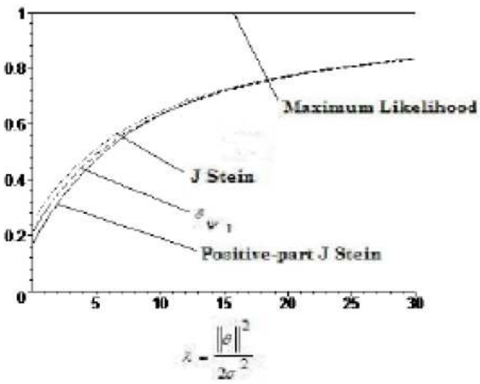

Fig. 2. Graph of risk ratios $\frac{R(\delta_{\psi_1},\theta)}{R(X,\theta)}, \frac{R(\delta^{JS},\theta)}{R(X,\theta)}, \frac{R(\delta^{JS+},\theta)}{R(X,\theta)}$ as function of $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ for $n = 30$ and $p = 10$

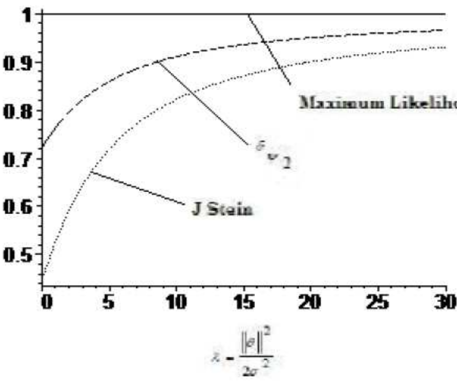

Fig. 3. Graph of risk ratios $\frac{R(\delta_{\psi_2},\theta)}{R(X,\theta)}, \frac{R(\delta^{JS},\theta)}{R(X,\theta)}$ as function of $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ for $n = 10$ and $p = 6$

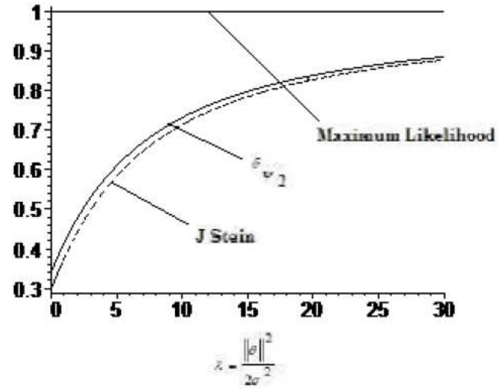

Fig. 4. Graph of risk ratios $\frac{R(\delta_{\psi_2},\theta)}{R(X,\theta)}, \frac{R(\delta^{JS},\theta)}{R(X,\theta)}$ as function of $\lambda = \frac{\|\theta\|^2}{2\sigma^2}$ for $n = 30$ and $p = 8$

Conclusion

In context of the study of asymptotic behavior of the risk ratios of shrinkage estimators of the mean θ of a multivariate Gaussian random $X \sim N_p(\theta, \sigma^2 I_p)$ in $\Re^p$, Casella and Hwang (1982) showed that if $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p} = c > 0$ then the ratios $\frac{R(\delta^{JS},\theta)}{R(X,\theta)}$ and $\frac{R(\delta^{JS+},\theta)}{R(X,\theta)}$ tend to $\frac{c}{1+c}$, thus the James-Stein estimator δ^{JS} and the positive-part of James-Stein estimator δ^{JS+}, which are minimax estimators, dominating the maximum likelihood estimator X if the dimension of parameter space p tends to infinity. In our work by taking the same model, namely $X \sim N_p(\theta, \sigma^2 I_p)$ with σ^2 is unknown and estimated by the statistic $S^2 \sim \sigma^2 \chi^2_n$ independent of X, we showed that for the shrinkage estimators of the form $\delta = \left(1 - \psi(S^2, \|X^2\|)\frac{S^2}{\|X^2\|}\right)X$, which did not inevitably minimax, we obtain the same ratio $\frac{c}{1+c}$ constant which is less than 1, when n and p tend simultaneously to infinity without assuming any order relation or functional relation between n and p, provided $\lim_{p \to +\infty} \frac{\|\theta\|^2}{p\sigma^2} = c$.

An idea would be to see whether one can obtain similar results of the asymptotic behaviour of risk ratios in the general case of the symmetrical spherical models, for general classes of shrinkage estimators. Expanding our work to minimax estimators proposed by Maruyama (2014) is also an idea that we currently explore.

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Author's Contributions

Both authors participated in doing the research and writing the paper.

Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

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