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A New Sequence Space Defined by Spectrum Operator and Musielak-Orlicz Function

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Abstract: In this study using the concept of generalized difference operator we have introduced and examined various spectrum of the operator $D(p, q, r, s, t, u)$ on the sequence space χ^3 defined by Musielak-Orlicz function. Moreover, we have established some relations concerning with this space.

Keywords: Analytic Sequence, Museiak-Orlicz Function, Triple Sequences, Chi Sequence, Fine Spectrum, Approximate Point Spectrum, Defect Spectrum and Compression Spectrum
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Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol (1978) and double sequence spaces is found in Hardy (1917; Deepmala and Mishra, 2016; Deepmala *et al*., 2016) and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner *et al*. (2007; Esi, 2014; Esi and Necdet Catalbas, 2014; Esi and Savas, 2015; Subramanian and Esi, 2015; Prakash *et al*., 2016) and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where:

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{i,jq} \left(m,n,k = 1,2,3,\ldots\right)$$

A sequence $x = (x_{mnk})$ is said to be triple analytic if:

$$\sup_{m,n,k} \left|x_{mnk}\right|^{\frac{1}{m+n+k}} < \infty$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if:

$$\left|x_{mnk}\right|^{\frac{1}{m+n+k}} \to 0 \; as \, m,n,k \to \infty$$

The vector space of all triple entire sequences are usually denoted by Γ^3: The spaces Λ^3 and Γ^3 are metric spaces with the metric:

$$d(x,y) = \sup_{m,n,k} \left\{ \left|x_{mnk} - y_{mnk}\right|^{\frac{1}{m+n+k}} : m,n,k : 1,2,3,\ldots \right\} \quad (1.1)$$

For all $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in Γ^3: Let $\phi = \{$finite sequences$\}$.

Consider a triple sequence x = (x_{mnk}), The $(m, n, k)^{\text{th}}$ section $x[m, n, k]$ of the sequence is defined by $x[m, n, k] = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \Im_{ijq}$ for all $m, n, k \in \mathbb{N}$:

$$\Im_{ijq} = \begin{bmatrix} 0 & 0 & \ldots 0 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & \ldots 1 & 0 & \ldots \\ 0 & 0 & \ldots 0 & 0 & \ldots \\ . & . & \ldots. & . & \ldots \\ . & . & \ldots. & . & \ldots. \end{bmatrix}$$

with 1 in the $(i, j, q)^{\text{th}}$ position and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple gai sequence if $\left((m+n+k)!\left|x_{mnk}\right|\right)^{\frac{1}{m+n+k}} \to 0$ as m, n, $k\to 1$. The triple gai sequences will be denoted by χ^3. The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz (1981) as follows:

$$Z(\Delta) = \left\{x = \left(x_k\right) \in w : \left(\Delta x_k\right) \in Z\right\}$$

for $Z = c$, c_0 and ℓ_∞, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_∞ denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \le p \le \infty$ 1 by Basar and Altay and in the case $0 < p < 1$ by Altay and Basar. The spaces $c(\Delta)$, c_0 (Δ), ℓ_∞ (Δ) and bv_p are Banach spaces normed by:

$$\| x \| = | x_1 | + \sup_{k\ge 1} | \Delta x_k | \, and \, \| x \|_{bv_p}$$
$$= \left(\sum_{k=1}^{\infty} \| x_k |^p\right)^{1/p}, (1 \le p \le \infty)$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by:

$$Z(\Delta) = \left\{x = \left(x_{mn}\right) \in w^2 : \left(\Delta x_{mn}\right) \in Z\right\}$$

where, $Z = \Lambda^2$, χ^2 and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all m, $n \in \mathbb{N}$. The generalized difference double notion has the following representation: $\Delta^m x_{mn} = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{mn+1} - \Delta^{m-1} x_{m+1n} + \Delta^{m-1} x_{m+1n+1}$ and also this generalized difference double notion has the following binomial representation: $\Delta^m x_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{m}(-1)^{i+j}\binom{m}{i}\binom{m}{j} x_{m+i,n+j}$

Let w^3, χ^3 (Δ_{mnk}) and Λ^3 (Δ_{mnk}) be denote the spaces of all, triple gai difference sequence space and triple analytic difference sequence space respectively and is defined as:

$$\Delta_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k}$$
$$+ x_{m+1,n+1,k} + x_{m+1,n,k+1} - x_{m+1,n+1,k+1} \; and \; \Delta^0 x_{mnk} = \left\langle x_{mnk}\right\rangle$$

Definitions and Preliminaries

Definition (Kamthan and Gupta, 1981)

An Orlicz function is a function M: $[0,\infty)\to[0,\to)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x)>0$, for $x>0$ and $M(x)\to\infty$ as $x\to\infty$. If convexity of Orlicz function M is replaced by $M(x + y) \le M(x) + M(y)$; then this function is called modulus function.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct Orlicz sequence space:

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{| x_k |}{\rho}\right) < \infty, for\, some\, \rho > 0\right\}$$

The space ℓ_M with the norm:

$$\| x \| = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{| x_k |}{\rho}\right) \le 1\right\}$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p$ $(1\le p<\infty 1)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p,

A sequence $f = (f_{mnk})$ of Orlicz function is called a Musielak-Orlicz function (Musielak, 1983). A sequence $g = (g_{mn})$ defined by:

$$g_{mn}(\upsilon) = \sup\left\{| \upsilon | u - \left(f_{mnk}\right)(u) : u \ge 0\right\}, m,n,k = 1,2,...$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows:

$$t_f = \left\{x \in w^3 : I_f\left(| x_{mnk} |\right)^{1/m+n+k} \to 0 as\, m,n,k \to \infty\right\}$$

where, I_f is a convex modular defined by:

$$I_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(| x_{mnk} |\right)^{1/m+n+k}, x = \left(x_{mnk}\right) \in t_f$$

We consider t_f equipped with the Luxemburg metric:

$$d(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{| x_{mnk} |^{1/m+n+k}}{mnk}\right)$$

is an exteneded real number.

Let X and Y be Banach metric spaces and T: $X\to Y$ be a bounded linear operator. The set of all bounded linear operators on X into itself is denoted by $B(X)$. The adjoint T^*: $X^*\to X^*$ of T is defined by $(T^*\phi)(x) = \phi(Tx)$ for all $\phi\in X^*$ and $x\in X$. Clearly, T^* is a bounded linear operator on the dual space X^*.

Let T: $D(T)\to X$ a linear operator, defined on $D(T)\subset X$, where $D(T)$ denote the domain of T and X is a complex normed linear space. For $T\in B(X)$ we associate a complex number α with the operator $(T-\alpha I)$ denoted by T_α defined on the same domain $D(T)$, where I is the identity operator. The inverse $(T-\alpha I)^{-1}$, denoted by T_α^{-1} is known as the resolvent operator of T. Many properties of T_α and T_α^{-1} depend on á and spectral theory is concerned with those properties. We are interested in the set of all α in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another essential property. We also detemine α's, for which the domain of T_α^{-1} is dense in X.

A regular value is a complex number α of T such that:

- (N_1) T_α^{-1} exists
- (N_2) T_α^{-1} is bounded and
- (N_3) T_α^{-1} is defined on a set which is dense in X

The resolvent set of T is the set of all such regular values α of T, denoted by $\rho(T)$. Its complement is given by $C\backslash\rho(T)$ in the complex plane C is called the spectrum of T, denoted by $\sigma(T)$. Thus the spectrum $\sigma(T)$ consists of those values of $\alpha\in C$, for which T_α is not invertible.

We discuss about the point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular in quantum mechanics.

Definition

The point (discrete) spectrum σ_p (T,X) is the set of complex number α such that T_α^{-1} does not exist. Further $\alpha\in p$ (T,X) is called the eigen value of T.

Definition

The continuous spectrum σ_c (T,X) is the set of complex number α such that T_α^{-1} exists and satisfies (N_3) but not (N_2) that is T_α^{-1} is unbounded.

Definition

The residual spectrum σ_r (T,X) is the set of complex number α such that T_α^{-1} exists (and may be bounded or not) but not satisfy (N_3), that is the domain of T_α^{-1} is not dense in X.

This is to note that in finite dimensional case, continuous spectrum coincides with the residual spectrum and equal to the empty set and the spectrum consists of only the point spectrum.

Given a bounded linear operator T in a Banach metric space X, we call a sequence $(x_{mnk})\in X$ as a sequence for T if:

$$d(x,0)=1 \Rightarrow \| x_{mnk}-0 \| = 1 = \| x_{mnk} \| \tag{2.1}$$

and:

$$d(Tx,0)=1 \Rightarrow \left\| Tx_{mnk}-0 \right\| = \left\| Tx_{mnk} \right\| \to 0 \, as \, m,n,k \to \infty \tag{2.2}$$

Definition

The approximate point spectrum σ_{ap} (T,X) = $\{\alpha\in C$: there exists (2.1), (2.2) sequence for T-$\alpha I\}$.

Definition

The defect spectrum σ_d (T,X) = $\{\alpha\in C$:T-αI is not subjective$\}$.

Definition

The compression spectrum σ_d (T,X) = $\left\{\alpha\in C : \overline{R(T-\alpha I)} \neq X\right\}$

Goldberg's Classification of Spectrum (Paul and Tripathy, 2016; Goldberg, 1985)

If X is Banach metric space and $T\in B(X)$; then there are three possibilities for $R(T)$:

- $R(T) = X$
- $R(T) \neq \overline{R(T)} = X$
- (III) $\overline{R(T)} \neq X$
  - T^{-1} exists and is continuous
  - T^{-1} exists but is discontinuous
  - T^{-1} does not exist

Definition (Musielak, 1983)

Let $n\in\mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1,..., x_n)$ = $\|(d_1(x_1, 0),..., d_n(x_n, 0))\|_p$ on X satisfying the following four conditions:

- $\|(d_1(x_1, 0),..., d_n(x_n, 0))\|_p = 0$ if and only if $d_1(x_1, 0),..., d_n(x_n, 0)$ are linearly dependent
- $\|(d_1(x_1, 0),..., d_n(x_n, 0))\|_p$ is invariant under permutation
- $\|(d_1(x_1, 0),..., \alpha d_n(x_n, 0))\|_p = |\alpha| \, \|(d_1(x_1, 0),..., d_n(x_n, 0))\|_p$, $\alpha\in R$
- d_p $((x_1, y_1), (x_2, y_2)\cdots(x_n, y_n)) = (d_X(x_1, x_2,\cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p}$ for $1\leq p <\infty$
- d $((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) :=$ sup $\{d_X(x_1, x_2,\cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$, for $x_1, x_2,\cdots x_n\in X$, $y_1, y_2,\cdots y_n\in Y$ is called the p product metric

Definition

Let $A = \left(a_{k,\ell}^{mn}\right)$ denote a four dimensional summability method that maps the complex triple sequences x into the triple sequence Ax where the k, ℓ-th term of Ax is as follows:

$$(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{k\ell}^{mn} x_{mnk}$$

such transformation is said to be non-negative if $a_{k\ell}^{mn}$ is non-negative.

Let E and F be two sequence spaces and $A = \left(a_{k,\ell}^{mn}\right)$ be an four dimensional infinite matrix of real or complex numbers $a_{k,\ell}^{mn}$, where $m, n, k \in \mathbb{N}$. Then A: $E \to F$, if for every sequence $x = (x_{mnk})_{k\ell} \in E$ the sequence $Ax = \{(Ax)_{k\ell}\}$ is in F where $(Ax)_{k\ell} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_{k\ell}^{mn} x_{mnk}$, provided the right hand side converges for every k, $\ell \in \mathbb{N}$ and $x \in E$.

Consider the operator $D(p, q, r, s, t, u)$, where:

$$D(p,q,r,s) = \begin{pmatrix} p & 0 & 0 & 0 & 0 & 0 & 0 & 0 & . & . \\ q & p & 0 & 0 & 0 & 0 & 0 & 0 & . & . \\ r & q & p & 0 & 0 & 0 & 0 & 0 & . & . \\ s & r & q & p & 0 & 0 & 0 & 0 & . & . \\ t & s & r & q & p & 0 & 0 & 0 & . & . \\ u & t & s & r & q & p & 0 & 0 & . & . \\ 0 & u & t & s & r & q & p & 0 & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & \ddots & & \end{pmatrix}$$

Remark

In particular if we consider $p = 1$, $q = 1$, $r = 1$, $s = 1$, $t = 1$, $u = 1$ then $D(p, q, r, s, t) = \Delta_3$.

Definition

Let f be an sequence of Musielak-Orlicz functions and a sequence of spectrum operator h is defined as following:

$$\begin{bmatrix} \chi_f^3\left(\sigma\left(D(p,q,r,s,t)\right)\right), \\ \|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\|_p \end{bmatrix} = \lim_{m,n,k\to\infty} \left[f\begin{pmatrix} \sigma\left(D(p,q,r,s,t)\right)\left((m+n+k)!\,|x_{mnk}|\right)^{(1/m)+nk} \\ ,\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\|_p \end{pmatrix} \right] = 0$$

$$\left[\Lambda_f^3\left(\sigma\left(D(p,q,r,s,t)\right)\right), \|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\|_p\right]$$
$$= \sup_{mnk}\left\{ \left[f\begin{pmatrix} \sigma\left(D(p,q,r,s,t)\right)|x_{mnk}|^{(1/m)+nk}, \\ \|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\|_p \end{pmatrix} \right] < \infty \right\}$$

Main Results

Theorem

If $\alpha,\beta,\mu,\eta, \quad = p, q, r, s, t$ then $\alpha,\beta, \gamma, \mu, \eta \in III_1[\chi_f^3(\sigma(D(p, q, r, s, t, u))), \|(d(x_1, 0), d(x_2, 0),\cdots, d(x_{n-1}, 0))\|_p)$.

Proof

If $\alpha = p$, $\beta = q$, $\gamma = r$, $\mu = s$, $\eta = t$ then the operator $D(p, q, r, s, t, u)$-αI-βI-γI-μI-$\eta I = D(0, 0, 0, 0, 0, u)$. Since $R(D(0, 0, 0, 0, 0, u)) \neq [\chi_f^3(\sigma(D(p, q, r, s, t, u))), k(d(x_1, 0), d(x_2, 0),\cdots, d(x_{n-1}, 0))\|_p]$. It is not invertible and hence $[\chi_f^3(\sigma(D(p, q, r, s, t, u))), k(d(x_1, 0), d(x_2, 0),\cdots, d(x_{n-1}, 0))\|_p] \in III_1$.

Therefore we have $\|[\chi_f^3(\sigma(D(p, q, r, s, t, u))), \|(d(x_1, 0), d(x_2, 0),\cdots, d(x_{n-1}, 0))\|_p]\| = \frac{u}{3} d(x, 0)$. It is bounded below and it has a bounded inverse. Hence α, β, γ, μ, $\eta \in III_1[\chi_f^3(\sigma(D(p, q, r, s, t, u))), \|(d(x_1, 0), d(x_2, 0),\cdots, d(x_{n-1}, 0))\|_p]$. This completes the proof.

Lemma:

$$\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right), \left\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right)$$
$$= \{\alpha,\beta,\gamma,\mu,\eta \in \mathbb{C} : |\alpha - p|, |\beta - q|, |\gamma - r\| \mu - s|, |\eta - t| \leq |u|\}$$

Theorem:

$$\begin{bmatrix} \chi_f^3\left(\sigma_{ap}\left(D(p,q,r,s,t,u)\right)\right), \\ \left\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p \end{bmatrix}$$
$$= \begin{Bmatrix} \alpha,\beta,\gamma,\mu,\eta \in \mathbb{C} : |\alpha - p|, |\beta - q|, \\ |\gamma - r|, |\mu - s|, |\eta - t| \leq |u| \end{Bmatrix} \setminus \{p\},\{q\},\{r\},\{s\},\{t\}$$

Proof

We have:

$$\left[\chi_f^3\left(\sigma_{ap}\left(D(p,q,r,s,t,u)\right)\right), \left\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right)$$
$$= \left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right), \left\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right) \setminus$$
$$III_1\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right), \left\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right),$$
$$\left[\chi_f^3\left(\sigma_{ap}\left(D(p,q,r,s,t,u)\right)\right), \left\|\left(d(x_1,0), d(x_2,0), \cdots, d(x_{n-1},0)\right)\right\|_p\right)$$
$$= \{\alpha,\beta,\gamma,\mu,\eta \in \mathbb{C} : |\alpha - p|, |\beta - q|, |\gamma - r|, |\mu - s|, |\eta - t| \leq |u|\} \setminus$$
$$\{p\},\{q\},\{r\},\{s\},\{t\}$$

is obtained by Lemma (3.2) and Theorem (3.1). This completes the proof.

Lemma:

$$\left[\begin{array}{l}\chi_f^3\left(\sigma_p\left(D(p,q,r,s,t,u)\right)\right),\\ \left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\end{array}\right) = \phi$$

Theorem:

$$\left[\chi_f^3\left(\sigma_d\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\{\alpha,\beta,\gamma,\mu,\eta\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|,|\mu-s|,|\eta-t|\leq|u|\}\setminus$$
$$\{p\},\{q\},\{r\},\{s\},\{t\}$$

Proof

We have:

$$\left[\chi_f^3\left(\sigma_d\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\setminus$$
$$I_3\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$

Now:

$$I_3\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=II_3\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=III_3\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\left[\chi_f^3\left(\sigma_p\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=\phi$$

by Lemma (3.4). Hence:

$$\left[\chi_f^3\left(\sigma_d\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$

Lemma:

$$\left[\chi_f^3\left(\sigma_r\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\{\alpha,\beta,\gamma,\mu,\eta\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|,|\mu-s|,|\eta-t|\leq|u|\}$$

Theorem:

$$\left[\chi_f^3\left(\sigma_{co}\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\{\alpha,\beta,\gamma,\mu,\eta\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|,|\mu-s|,|\eta-t|\leq|s|\}\setminus$$
$$\{p\},\{q\},\{r\},\{s\},\{t\}$$

Proof:

$$\left[\chi_f^3\left(\sigma_{c_0}\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=III_1\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\cup$$
$$III_2\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\cup$$
$$III_3\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$

Now:

$$III_1\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)\cup$$
$$III_2\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\left[\chi_f^3\left(\sigma_r\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\{\alpha,\beta,\gamma,\mu,\eta\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|,|\mu-s|,|\eta-t|\leq|u|\}$$

is obtained by Lemma (3.6). Again:

$$III_3\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\left[\chi_f^3\left(\sigma_p\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)=\phi$$

is obtained by Lemma (3.4). Hence:

$$\left[\chi_f^3\left(\sigma_{c_0}\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d\left(x_1,0\right),d\left(x_2,0\right),\cdots,d\left(x_{n-1},0\right)\right)\right\|_p\right)$$
$$=\{\alpha,\beta,\gamma,\mu,\eta\in\mathbb{C}:|\alpha-p|,|\beta-q|,|\gamma-r|,|\mu-s|,|\eta-t|\leq|u|\}\setminus$$
$$\{p\},\{q\},\{r\},\{s\},\{t\}$$

This completes the proof.

Lemma

The adjoint operator T^* of T is onto if and only if T has a bounded inverse.

Theorem

If α, β, γ, μ, $\eta = p$, q, r, s, t, u then:

$$\alpha,\beta,\gamma,\mu,\eta \in III_1\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)$$

Proof

By Lemma (3.6):

$$\alpha,\beta,\gamma,\mu,\eta \in III_1\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)$$

whenever $\alpha = p$, $\beta = q$, $\gamma = r$, $\mu = s$, $\eta = t$. By Lemma (3.8), $\alpha = p$, $\beta = q$, $\gamma = r$, $\mu = s$, $\eta = t$ is not in $\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)$ and hence:

$$\left(\begin{array}{l}\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right) \\ -\alpha I,\beta I,\gamma I,\mu I,\eta I\end{array}\right)^{-1}$$

exists. We have to prove that:

$$\left(\begin{array}{l}\left[\chi_f^3\left(\sigma\left(D(p,q,r,s,t,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right) \\ -\alpha I,\beta I,\gamma I,\mu I,\eta I\end{array}\right)^{-1}$$

must be continuous, hence it is show that:

$$\begin{array}{l}\left[\chi_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)^* \\ -\alpha I,\beta I,\gamma I,\mu I,\eta I \\ =\left[\chi_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)^*\end{array}$$

is onto by Lemma (3.8). Given:

$$y = (y_{mnk}) \in \left[\begin{array}{l}\Lambda_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right), \\ \left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\end{array}\right)$$

we can find:

$$x = (x_{mnk}) \in \left[\begin{array}{l}\Lambda_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right), \\ \left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\end{array}\right)$$

such that:

$$\begin{array}{l}\left[\chi_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right) \\ =\left[\Lambda_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)\end{array}$$

Therefore we have:

$$x_{mnk} = \frac{1}{u} y_{m-1,n-2,k-3}$$

which shows that:

$$\left[\chi_f^3\left(\sigma\left(D(0,0,0,0,0,u)\right)\right),\left\|\left(d(x_1,0),d(x_2,0),\cdots,d(x_{n-1},0)\right)\right\|_p\right)^*$$

is onto. This completes the proof.

Conclusion

Author's are introduced and examined various spectrum of the operator $D(p, q, r, s, t, u)$ on the sequence space χ^3 defined by Musielak-Orlicz function.

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Author's Contributions

First author study using the concept of generalized difference operator are introduced and second author examined various spectrum of the operator $D(p, q, r, s, t, u)$ on the sequence space χ^3 defined by Musielak-Orlicz function. Moreover, we have established some relations concerning with this space.

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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