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Hilbert-Type Inequalities Revisited

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Abstract: Considering different parameters, Hilbert-type integral inequality for functions $f(x)$, $g(x)$ in $L^2[0, \infty)$ will be generalized.

Keywords: Hilbert Inequality, Cauchy Inequality, Beta Function

Introduction

We establish more general variants of the integral Hilbert-type inequality (Hardy *et al*., 1934):

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy \le \frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)dx\right)^{1/p}\left(\int_0^\infty g^q(x)dx\right), \tag{1}$$

unless $f(x) \equiv 0$ or $g(x) \equiv 0$, where $p>1$, $q = p/(p-1)$. Inequality (1), would be invalid for some $f(x)$, $g(x)$ if the constant π cosec (π/p) were replaced by a smaller number see (Hardy *et al*., 1934). Inequality (1) with its modifications have played an important role in the raise of many mathematical and physical branches see for instance (Xingdong and Bicheng, 2010; Jichang and Debnath, 2000).

In this study we are concerned with the case when $p = q = 2$, i.e., we focus on the inequality:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy \le \pi\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx\right)^{1/2}, \tag{2}$$
$$f(x), g(x) \in L^2[0,\infty).$$

Many mathematicians have worked on generalizing inequality (2) in different ways. Some of them developed half discrete analogues of (2) see for instance (Xin and Yang, 2012; Zhenxiao and Yang, 2013), while others worked on developing different variants of the denominator of the left hand side see for example (Bicheng, 1998; Bicheng and Qiang, 2015; Bing *et al*., 2015; Jichang and Debnath, 2000). For example in (Bicheng, 1998) the following inequality can be found: for $0 < a < b$ and $0 < \lambda \le 1$, $f(x)$, $g(x) \in L^2[0, \infty)$ we have:

$$\int_a^b\int_a^b \frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy \le \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left(1-\left(\frac{a}{b}\right)^{\lambda/4}\right)\left(\int_a^b x^{1-\lambda}f^2(x)dx\int_a^b x^{1-\lambda}g^2(x)dx\right)^{1/2}. \tag{3}$$

The objective of this paper is to derive more general form of Hilbert's inequality (2) by introducing some parameters. In particular we generalize inequality (3) focusing on developing the denominator of the left hand side. In this study $\beta(p, q)$ is the β-function.

Main Results and Discussion

This section states and discusses the main theorem which will be proved in the fourth section. For different parameters t, $\lambda \in (0, 1]$ we have the following theorem.

Theorem 2.1

Suppose that $0 < a < b$, $0 < c < d$, A, B are nonzero real numbers and $0 < t$, $\lambda \le 1$. Then for functions $f(x)$, $g(x) \in L^2[0,\infty)$ the following Hilbert-type inequality holds:

$$\int_a^b\int_c^d \frac{f(x)g(y)}{(Ax^t+By^t)^\lambda}dx\,dy \le \frac{\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{t(AB)^{\frac{\lambda}{2}}}\left[\left(1-\left(\frac{a}{b}\right)^{t\lambda/4}\right)\left(1-\left(\frac{c}{d}\right)^{t\lambda/4}\right)\int_c^d x^{1-t\lambda}f^2(x)dx\int_a^b x^{1-t\lambda}g^2(x)dx\right]^{1/2}. \tag{4}$$

Remark 2.2

If $A = B = 1$ and $t = 1$, Theorem 2.1 gives:

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$$\int_a^b\int_c^d \frac{f(x)g(y)}{(x+y)^\lambda}dx\,dy \le \beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\left[\left(1-\left(\frac{a}{b}\right)^{\lambda/4}\right)\left(1-\left(\frac{c}{d}\right)^{\lambda/4}\right) \int_c^d x^{1-\lambda}f^2(x)dx\int_a^b x^{1-\lambda}g^2(x)dx\right]^{1/2}. \quad (5)$$

Remark 2.3

If $a = c$ and $b = d$, then inequality (5) reduces to inequality (3), which in turn leads to the original Hilbert's inequality (2) if $\lambda = 1$ and $a \to 0$ and $b \to \infty$.

To prove Theorem 2.1, we prove first two lemmas introduced in the following section.

Lemmas

In this section we present and prove two needed lemmas.

Lemma 3.1

For parameters t, λ where $0 < t, \lambda \le 1$, define $\phi_{t,\lambda}$ and $\psi_{t,\lambda}$ as:

$$\phi_{t,\lambda} := \int_0^\infty \frac{1}{(1+u^t)^\lambda}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du,\ \psi_{t,\lambda} := \int_0^1 \frac{1}{(1+u^t)^\lambda}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du. \quad (6)$$

Then $\phi_{t,\lambda} = \frac{1}{t}\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ and $\psi_{t,\lambda} = \frac{1}{2}\phi_{t,\lambda}$.

Proof

Put $u^t = v$ in $\psi_{t,\lambda}$ to obtain:

$$\psi_{t,\lambda} = \frac{1}{t}\int_0^1 \frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{1-\frac{t\lambda}{2}}dv.$$

However, it is known that the Beta function is given by (see for instance (Greene and Krantz, 2006)):

$$\beta(p,q) = \int_0^1 \frac{y^{p-1}+y^{q-1}}{(1+y)^{p+q}}dy,$$

which produces:

$$\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) = 2\int_0^1 \frac{1}{(1+y)^\lambda}\left(\frac{1}{y}\right)^{1-\frac{\lambda}{2}}dy.$$

Therefore:

$$\psi_{t,\lambda} = \frac{1}{2t}\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right) \quad (7)$$

Now, substituting $u^t = v$ in $\phi_{t,\lambda}$ gives:

$$\begin{aligned}\phi_{t,\lambda} &= \frac{1}{t}\left(\int_0^1 \frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{1-\frac{\lambda}{2}}dv + \int_1^\infty \frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{1-\frac{\lambda}{2}}dv\right)\\ &= \psi_{t,\lambda} + \frac{1}{t}\int_1^\infty \frac{1}{(1+v)^\lambda}\left(\frac{1}{v}\right)^{1-\frac{\lambda}{2}}dv\\ &= \psi_{t,\lambda} + \frac{1}{t}\int_0^1 \frac{1}{\left(1+\frac{1}{y}\right)^\lambda}(y)^{1-\frac{\lambda}{2}-2}dy\\ &= \psi_{t,\lambda} + \frac{1}{t}\int_0^1 \frac{1}{(1+y)^\lambda}\left(\frac{1}{y}\right)^{1-\frac{\lambda}{2}}dy = 2\psi_{t,\lambda}.\end{aligned} \quad (8)$$

Hence, from (7) and (8) we obtain $\phi_{t,\lambda} = \frac{1}{t}\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)$ as stated above.

Lemma 3.2

For parameters $t, \lambda \in (0, 1]$, define $h_{t,\lambda}(y)$ as:

$$h_{t,\lambda}(y) := y^{-\frac{t\lambda}{2}}\int_0^y \frac{1}{(1+u^t)^\lambda}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du,\ y \in (0,1]. \quad (9)$$

Then $h_{t,\lambda}(y) \ge h_{t,\lambda}(1) = \psi_{t,\lambda}$ (defined by (6)). The equality holds when $y = 1$.

Proof

For $y \in (0, 1]$ we have:

$$\frac{d}{dy}h_{t,\lambda}(y) = -y^{-1-\frac{t\lambda}{2}}\int_0^y \frac{1}{(1+u^t)^\lambda}du^{\frac{t\lambda}{2}} + y^{-1}\frac{1}{(1+y^t)^\lambda}.$$

Integrating the first term by parts leads to:

$$\frac{d}{dy}h_{t,\lambda}(y) = -t\lambda y^{-1-\frac{t\lambda}{2}}\int_0^y \frac{u^{\frac{t\lambda}{2}+t-1}}{(1+u^t)^{\lambda+1}}du < 0.$$

Therefore, $h_{t,\lambda}(y)$ is strictly decreasing on (0, 1]. Hence $h_{t,\lambda}(y) \ge h_{t,\lambda}(1) = \psi_{t,\lambda}$. As required.

We use Lemmas 3.1 and 3.2 to prove our main result.

Proofs of Main Results

Proof of Theorem 2.1

By Cauchy's inequality, we can estimate the left hand side of (4) as follows:

$$\int_a^b\int_c^d \frac{f(x)g(y)}{\left(Ax^t+By^t\right)^{\lambda}}dx\,dy=\int_a^b\int_c^d \frac{f(x)}{\left(Ax^t+By^t\right)^{\lambda/2}}\left(\frac{x}{y}\right)^{(1-t\lambda/2)/2}$$
$$\frac{g(y)}{\left(Ax^t+By^t\right)^{\lambda/2}}\left(\frac{y}{x}\right)^{(1-t\lambda/2)/2}dx\,dy$$
$$\leq\left(\begin{array}{l}\int_a^b\int_c^d \frac{f^2(x)}{\left(Ax^t+By^t\right)^{\lambda}}\left(\frac{x}{y}\right)^{1-t\lambda/2}dx\,dy\\ \int_a^b\int_c^d \frac{g^2(y)}{\left(Ax^t+By^t\right)^{\lambda}}\left(\frac{y}{x}\right)^{1-t\lambda/2}dx\,dy\end{array}\right)^{1/2} \quad (10)$$
$$=\left(\int_c^d w_{t,\lambda}(a,b,x)f^2(x)dx\int_a^b w_{t,\lambda}(c,d,y)g^2(y)dy\right)^{1/2},$$

where:

$$w_{t,\lambda}(a,b,x):=\int_a^b\frac{1}{\left(Ax^t+By^t\right)^{\lambda}}\left(\frac{x}{y}\right)^{1-\frac{t\lambda}{2}}dy, \quad (11)$$

and:

$$w_{t,\lambda}(c,d,y):=\int_c^d\frac{1}{\left(Ax^t+By^t\right)^{\lambda}}\left(\frac{y}{x}\right)^{1-\frac{t\lambda}{2}}dy. \quad (12)$$

Substituting $u=\left(\frac{B}{A}\right)^{\frac{1}{t}}\frac{y}{x}$ in (11) leads to:

$$w_{t,\lambda}(a,b,x)=\frac{x^{1-t\lambda}}{(AB)^{\lambda/2}}\left(\begin{array}{l}\phi_{t,\lambda}-\int_0^{(B/A)^{\frac{1}{t}}\frac{a}{x}}\frac{1}{\left(1+u^t\right)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du\\ -\int_{(B/A)^{\frac{1}{t}}\frac{b}{x}}^{\infty}\frac{1}{\left(1+u^t\right)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du\end{array}\right).$$

Use the substitution $u=\frac{1}{v}$ in the second integral to have:

$$w_{t,\lambda}(a,b,x)=\frac{x^{1-t\lambda}}{(AB)^{\lambda/2}}\left(\phi_{t,\lambda}-\left(\begin{array}{l}\int_0^{(B/A)^{\frac{1}{t}}\frac{a}{x}}\frac{1}{\left(1+u^t\right)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du\\ +\int_0^{(A/B)^{\frac{1}{t}}\frac{x}{b}}\frac{1}{\left(1+u^t\right)^{\lambda}}\left(\frac{1}{u}\right)^{1-\frac{t\lambda}{2}}du\end{array}\right)\right) \quad (13)$$
$$=\frac{x^{1-t\lambda}}{(AB)^{\lambda/2}}\left\{\phi_{t,\lambda}-\left[\begin{array}{l}\left(\left(\frac{B}{A}\right)^{1/t}\frac{a}{x}\right)^{\frac{t\lambda}{2}}h_{t,\lambda}\left(\left(\frac{B}{A}\right)^{1/t}\frac{a}{x}\right)\\ +\left(\left(\frac{A}{B}\right)^{1/t}\frac{x}{b}\right)^{\frac{t\lambda}{2}}h_{t,\lambda}\left(\left(\frac{A}{B}\right)^{1/t}\frac{x}{b}\right)\end{array}\right]\right\},$$

where, $\phi_{t,\lambda}$ is as defined in Lemma 3.1 and $h_{t,\lambda}(.)$ is as defined in Lemma 3.2. Now, Apply Lemma 3.2 to equation (13) to obtain:

$$w_{t,\lambda}(a,b,x)\leq\frac{x^{1-t\lambda}}{(AB)^{\lambda/2}}\left\{\phi_{t,\lambda}-\psi_{t,\lambda}\left[\begin{array}{l}\left(\left(\frac{B}{A}\right)^{1/t}\frac{a}{x}\right)^{\frac{t\lambda}{2}}\\ +\left(\left(\frac{A}{B}\right)^{1/t}\frac{x}{b}\right)^{\frac{t\lambda}{2}}\end{array}\right]\right\} \quad (14)$$
$$\leq\frac{x^{1-t\lambda}}{(AB)^{\lambda/2}}\left\{\phi_{t,\lambda}-2\psi_{t,\lambda}\left[\left(\left(\frac{B}{A}\right)^{1/t}\frac{a}{x}\right)^{\frac{t\lambda}{2}}\cdot\left(\left(\frac{A}{B}\right)^{1/t}\frac{x}{b}\right)^{\frac{t\lambda}{2}}\right]^{1/2}\right\}.$$

Using Lemma 3.1 in (14) produces:

$$w_{t,\lambda}(a,b,x)\leq\frac{\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{t}\left(1-\left(\frac{a}{b}\right)^{\frac{t\lambda}{4}}\right)\frac{x^{1-t\lambda}}{(AB)^{\lambda/2}}. \quad (15)$$

Similarly, we can obtain:

$$w_{t,\lambda}(c,d,y)\leq\frac{\beta\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)}{t}\left(1-\left(\frac{c}{d}\right)^{\frac{t\lambda}{4}}\right)\frac{y^{1-t\lambda}}{(AB)^{\lambda/2}}. \quad (16)$$

Now substitute (15) and (16) into inequality (10) yields (4) as required.

Conclusion

We have derived new Hilbert-type inequality which can be considered as a generalization of previously proved ones.

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Ethics

This article is original and contains unpublished material. The corresponding author confirms that all of the other authors have read and approved the manuscript and no ethical issues involved.

References

Bicheng, Y. and C. Qiang, 2015. Two kinds of hilbert-type integral inequalities in the whole plane. J. Inequal. Applic., 2015: 21-21. DOI: 10.1186/s13660-014-0545-8

Bicheng, Y., 1998. Note on Hilbert's integral inequality. J. Math. Anal. Applic., 220: 778-778.

Bing, H., H. Xianyong, C. Junfei and Y. Bicheng, 2015. Hilbert-type and hardy-type integral inequalities with operator expressions and the best constants in the whole plane. J. Inequal. Applic., 2015: 129-129. DOI: 10.1186/s13660-015-0655-y

Greene, R.E. and S.G. Krantz, 2006. Function Theory of one Complex Variable. 1st Edn., American Mathematical Society, Providende, ISBN-10: 0821839624, pp: 504.

Hardy, G.H., J.E. Littlewood and G. Polya, 1934. Inequalities. 1st Edn., Cambridge University Press, Cambridge, pp: 314.

Jichang, K. and L. Debnath, 2000. On new generalizations of Hilbert's inequality and their applications. J. Math. Anal. Applic., 245: 248-265. DOI: 10.1006/jmaa.2000.6766

Xin, D. and B. Yang, 2012. A half-discrete Hilbert-type inequality with the non-monotone kernel and the best constant factor. J. Inequal. Applic., 2012: 184-184. DOI: 10.1186/1029-242X-2012-184

Xingdong, L. and Y. Bicheng, 2010. On a new hilbert-hardy-type integral operator and applications. J. Inequal. Applic., 2010: 812636-812636. DOI: 10.1155/2010/812636

Zhenxiao, H. and B. Yang, 2013. On a half-discrete hilbert-type inequality similar to mulholland's inequality. J. Inequal. Applied, 2013: 290-290. DOI: 10.1186/1029-242X-2013-290