Journal of Mathematics and Statistics 8 (4): 461-470, 2012
ISSN 1549-3644

doi:10.3844/jmssp.2012.461.470 Published Online 8 (4) 2012 (http://www.thescipub.com/jmss.toc)

Minimization of ℓ_2-Norm of the KSOR Operator

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Received 2012-08-17, Revised 2012-11-19; Accepted 2012-11-19

ABSTRAC

We consider the problem of minimizing the ℓ_2-norm of the KSOR operator when solving a linear systems of the form AX = b where, A = I +B (T_J = -B, is the Jacobi iteration matrix), B is skew symmetric matrix. Based on the eigenvalue functional relations given for the KSOR method, we find optimal values of the relaxation parameter which minimize the ℓ_2-norm of the KSOR operators. Use the Singular Value Decomposition (SVD) techniques to find an easy computable matrix unitary equivalent to the iteration matrix T_{KSOR}. The optimum value of the relaxation parameter in the KSOR method is accurately approximated through the minimization of the ℓ_2-norm of an associated matrix $\Delta(\omega^*)$ which has the same spectrum as the iteration matrix. Numerical example illustrating and confirming the theoretical relations are considered. Using SVD is an easy and effective approach in proving the eigenvalue functional relations and in determining the appropriate value of the relaxation parameter. All calculations are performed with the help of the computer algebra system "Mathematica 8.0".

Keywords: KSOR Iterative Method, ℓ_2-Norm, Singular Value Decomposition (SVD)

1. INTRODUCTION

We consider linear systems of the form Equation 1:

$$\sum_{j=1}^{m} a_{ij} x_j = b_j, a_{ii} \neq 0, i = 1,2,\cdots,m \tag{1}$$

With a_{ij} = - a_{ji} for i ≠ j and the system admits a unique solution. This system of equations can be written as Equation 2:

$$A\,X = b, AX,\ \ b \in R^m,\ A \in R^{m\times m} \tag{2}$$

Such linear systems arise in many different applications for example in the finite difference treatment of the Korteweg de Vries equation, Buckley (1977). Also, simi-lar linear systems appears in the treatment of coupled harmonic equations, Ehrlich (1972).

In the iterative treatment of linear systems, we use the splitting, A = D-L-U, where D = d×I_m is the diag-onal part of A, for some non-zero constant d, -L is the strictly lower-triangular part of A and -U is the strictly upper-triangular part of A, Woznicki (2001).

1.1. Jacobi Method Equation 3:

$$x_i^{[n+1]} = \frac{1}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{[n]} - \sum_{j=i+1}^{m} a_{ij} x_j^{[n]}\right) \tag{3}$$

The Jacobi Method in matrix form is Equation 4:

$$X^{[n+1]} = T_J X^{[n]} + D^{-1} b\, T_j = D^{-1}(L+U) \tag{4}$$

T_J is the Jacobi iteration matrix, it is clear that T_J in this case is a skew symmetric matrix.

1.2. The SOR Method is Equation 5:

$$x_i^{[n+1]} = x_i^{[n]} + \frac{\omega}{a_{ii}}\left(b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij} x_j^{[n+1]}}_{\text{updated components}} - \underbrace{\sum_{j=i+1}^{m} a_{ij} x_j^{[n]} - a_{ii} x_i^{[n]}}_{\text{old components}}\right) \tag{5}$$

$i = 1,2,\cdots,m;\ n = 0, 1,\cdots$

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where, $\omega\in(0,2)$ is a relaxation parameter, $\omega = 1$ gives the well-known Gauss-Seidel method.

The SOR Method in matrix form is Equation 6:

$$X^{[n+1]} = T_{SOR}X^{[n]} + (D-\omega L)^{-1}\omega b$$
$$T_{SOR} = (D-\omega L)^{-1}\left((1-\omega)D+\omega U\right) \tag{6}$$

where, T_{SOR} is the SOR iteration matrix.

The choice of the relaxation parameter ω is very important for the convergence rate of the SOR method. For certain classes of matrices (2-cyclic consistently ordered) with property A, in the sense of Young (2003), for such systems there is a functional eigenvalue relation of the form Equation 7:

$$\left(\lambda+\omega-1\right) = \omega\mu\lambda^{1/2} \tag{7}$$

where, λ is an eigenvalue of the T_{SOR} and μ is a corresponding eigenvalue of the T_J. Most work on the choice of ω is to minimize $\rho(T_{SOR})$ which is only an asymptotic criteria of the convergence rate of linear stationary iterative method, Hadjidimos and Neumann (1998). In real computations, we have to consider average convergence rate Milleo *et al.* (2006). The determination of the optimal value of the relaxation parameter ω_{opt} can be obtained with the help of the eigenvalue functional relation (7). Young (2003), determined ω_{opt} when T_J has only real eigenvalues and $\rho(T_J)\langle 1$. In this case we have:

$$\omega_{opt} = \frac{2}{1+\sqrt{1-(\rho(T_J))^2}}$$

where the optimality is understood in the sense of the minimization of $\rho(T_{SOR})$.

Maleev (2006) determined ω_{opt} when T_J has only pure imaginary eigenvalues and $\rho(T_J)\langle 1$. In this case we have:

$$\omega_{opt} = \frac{2}{1+\sqrt{1+(\rho(T_J))^2}}$$

Golub and Pillis (1990) introduced a simple proof for the eigenvalue functional relation (7) by the use of the Singular Value Decomposition (SVD) approach for real symmetric matrices. Yin and Yuan (2002) considered the skew symmetric case as well as the symmetric case. Milleo *et al.* (2006) considered the minimization of ℓ_2-norms of the SOR and MSOR operators for the skew symmetric case.

1.3. The KSOR Method is

In a recent work Youssef (2012), introduced the KSOR method Equation 8 and 9:

$$x_i^{[n+1]} = x_i^{[n]} + \frac{\omega^*}{a_{ii}}\left(b_i - \underbrace{\sum_{j=1}^{i-1} a_{ij}x_j^{[n+1]}}_{\text{updated}} - \underbrace{\sum_{j=i+1}^{m} a_{ij}x_j^{[n]}}_{\text{old}} - \underbrace{a_{ii}x_i^{[n+1]}}_{\text{Assumed updated}}\right) \tag{8}$$
$$i = 1,2,\cdots,m,\quad \omega^*\in R-[-2,0]$$

$$x_i^{[n+1]} = \frac{1}{(1+\omega^*)}\left(x_i^{[n]} + \frac{\omega^*}{a_{ii}}\left(b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{[n+1]}\sum_{j=i+1}^{m} a_{ij}x_j^{[n]}\right)\right) \tag{9}$$
$$i = 1,2,\cdots,m,\quad \omega^*\in R-[-2,0]$$

The KSOR Method in matrix notation is Equation 10:

$$X^{[n+1]} = T_{KSOR}X^{[n]} + \left((1+\omega^*)D-\omega^*L\right)^{-1}\omega^* b$$
$$T_{KSOR} = \left((1+\omega^*)D-\omega^*L\right)^{-1}[D+\omega^*U] \tag{10}$$

where, T_{KSOR} is the KSOR iteration matrix (operator).

As it was in the SOR the rate of convergence of the KSOR method depends on the choice of the relaxation parameter ω^*. For certain classes of matrices (2-cyclic consistently ordered with property A), Youssef (2012) established the eigenvalue functional relation Equation 11:

$$\left(\beta_i+\omega\beta_i-1\right) = \omega\,\mu_i\,\beta_i^{1/2} \tag{11}$$

where, β_I's are the eigenvalues of the T_{KSOR} and μ_i's are the eigenvalues of the Jacobi iteration matrix T_J. The eigenvalue functional relation (11) can be proved by the use of the SVD technique.

1.4. Singular Value Decoposition

Singular Value Decomposition (SVD) of a matrix $B\in R^{\ell\times m}$ is a factorization:

$$B = U\Sigma V^T, \Sigma = \text{diag}\left(s_1,s_2,\cdots,s_q\right)\in R^{\ell\times m}, q = \text{mim}\{\ell,m\}$$

where, $s_1 \geq s_2 \geq \ldots \geq s_q \geq 0$, U and V are orthogonal matrices such that:

$$U^TU = I_p, V^TV = I_q$$

We consider in this study the case studied by Yin and Yuan (2002) also by Milleo *et al*. (2006) in which the coefficient matrix take the form Equation 12:

$$A = \begin{pmatrix} I_p & -F \\ F^T & I_q \end{pmatrix} \quad (12)$$

where, $F \in \mathbb{R}^{p \times q}$ with $p + q = m$ and $p \geq q$.

In this case the Jacobi iteration matrix becomes Equation 13:

$$T_J = \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} \quad (13)$$

It is clear that T_J is skew symmetric and accordingly admits pure imaginary eigenvalues and the KSOR iteration matrix T_{KSOR} becomes Equation 14:

$$T_{KSOR} = \begin{pmatrix} \frac{1}{\omega^*+1} I_p & \frac{\omega^*}{\omega^*+1} F \\ \frac{-\omega^*}{(\omega^*+1)^2} F^T & \frac{1}{\omega^*+1} I_q - \frac{\omega^{*2}}{(\omega^*+1)^2} F^T F \end{pmatrix} \quad (14)$$

Usually, researchers work on obtaining the optimum value of the relaxation parameter in the sense of minimizing the spectral radius of the iteration matrix or an equivalent quantity. We use the SVD approach in proving the eigenvalue functional relation for the KSOR method. Also, we use the same argument of Golub and Pillis (1990) to define a matrix $\Delta(\omega^*)$ which has the same spectrum as the iteration matrix T_{KSOR}.

Our objective is to find the optimal value of the relaxation parameter ω^* which minimizes the ℓ_2-norm of the KSOR operator and illustrate the theoretical results through applications to a numerical example.

2. MATERIALS AND METHODS

We use the SVD in proving the relation between the eigenvalues of the skew symmetric Jacobi iteration matrix T_J and the singular values of a block sub-matrix F, theorem (1). We will prove the relation between the eigenvalue functional relation between the eigenvalues of T_J and T_{KSOR} by using SVD, theorem (2). We will find the spectal raduis of $((T_{KSOR})^T T_{KSOR})$, theorem (3). We will find the optimal value of the relaxation parameter ω^* to minimize the ℓ_2-norms of the T_{KSOR} theorem (4).

Theorem 1

Let Abe the matrix given by (12), then Equation 15:

$$\{\mu_i^2\} = \sigma(T_J^2) = (\sigma(T_J))^2 = \sigma(-FF^T) = \{-S_i^2\} i = 1,2...,q \quad (15)$$

where, μ_i^2 are the eigenvalues of T_J^2, S_i^2 are the squares of the singular values of F and $(\sigma(T_J))^2$ is the set of squares of the eigenvalues of T_J.

Proof

Using the SVD to decompose the corner block matrix F, we obtain Equation 16:

$$F = U\Sigma V^T \quad (16)$$

where, p×p matrix U and q×q matrix V are orthogonal, i.e.:

$$U^TU = I_p, V^TV = I_q$$

and Σ is the p×q diagonal matrix (of singular values) defined in (17). The eigenvalues of the matrix:

$FF^T = U\Sigma\Sigma^T U^T$ are $\{s_1^2, s_2^2, ..., s_q^2\}$

And:

$$FF^TU = U\Sigma\Sigma^TU^TU = U\Sigma\Sigma^T$$

Accordingly, the eigenvectors of the matrix FF^T equal to the columns of orthogonal matrix U. Similarly, $F^T F = V\Sigma^T \Sigma V^T$ has its eigenvectors equal to the columns of orthogonal matrix V. The number of nonzero singular values S_i of F is equal to the rank of F.

Substituting the singular value decomposition (16) into the corner elements F, F^T of (13), we obtain (18) Equation 17 and 18:

$$\Sigma = \begin{pmatrix} s_1 & 0 & \cdots & \cdots & 0 \\ 0 & s_2 & 0 & & 0 \\ \vdots & \ddots & \vdots & s_{q-1} & \vdots \\ 0 & \cdots & \cdots & \cdots & s_q \\ 0 & & \cdots & 0 & \\ \vdots & & \cdots & \vdots & \\ 0 & & \cdots & 0 & \end{pmatrix} \begin{matrix} \}q \times q \\ \\ \}(p-q) \times q \end{matrix} \quad (17)$$

$$T_J = \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & U\Sigma V^T \\ -V\Sigma^T U^T & 0 \end{pmatrix} \quad (18)$$

Now, we will find a relation between the singular values S_i (diagonal of Σ) and the eigenvalues μ_i of T_J where i = 1,2,..,q. For $\mu_i \neq 0$ an eigenvalues of T_J, we have Equation 19:

$$T_J\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \mu_i\begin{bmatrix} x_i \\ y_i \end{bmatrix} \text{iff } T_J\begin{bmatrix} x_i \\ -y_i \end{bmatrix} = -\mu_i\begin{bmatrix} x_i \\ -y_i \end{bmatrix} i = 1,2,\ldots,t \quad (19)$$

So that, the number of non-zero eigenvalues of T_J equals 2t that's come in pairs $\pm\mu_i$. To account for zero eigenvalues, we write Equation 20:

$$T_J\begin{bmatrix} z_i \\ z_i^* \end{bmatrix} = 0\begin{bmatrix} z_i \\ z_i^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad i = 1,2,\ldots,r \quad (20)$$

We construct the n×n non-singular matrix W whose columns are the orthogonal eigenvectors of (19) and (20):

$$W = \begin{bmatrix} X & X & Z \\ Y & -Y & Z^* \end{bmatrix} \quad n = p + q = 2t + r$$

Note that the t columns of p×t matrix X and q×t matrix Y are the t respective eigenvectors of (19), the r columns of p×r matrix Z and q×r matrix Z^T come from the r null vectors of (20).

Ordinarily, we would scale the columns of W to produce an orthogonal matrix as a technical convenience, however, we assume that the columns of W are scaled so that Equation 21:

$$WW^H = 2I \quad (21)$$

Let the matrix I denote the t×t matrix whose diagonal elements are the t positive eigenvalues μ_i of (19). Then (19) and (20) can be combined to produce the single matrix equation:

$$T_J\begin{bmatrix} X & X & Z \\ Y & -Y & Z^* \end{bmatrix} = \begin{bmatrix} X & X & Z \\ Y & -Y & Z^* \end{bmatrix}\begin{bmatrix} J & 0 & 0 \\ 0 & -J & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which, when multiplied through on the right by W^H we get Equation 22:

$$T_J = \begin{bmatrix} 0 & XJY^H \\ YJX^H & 0 \end{bmatrix} \quad (22)$$

Comparing the block entries of T_J in (18) and (22), we obtain the equalities:

$$F = XJY^H = U\Sigma V^T$$

And:

$$-F^T = YJX^H = -V\Sigma U^T$$

And we see Equation 23:

$$T_J^2 = \begin{bmatrix} XJ^2X^H & 0 \\ 0 & YJ^2Y^H \end{bmatrix} = \begin{bmatrix} -FF^T & 0 \\ 0 & -F^TF \end{bmatrix} = \begin{bmatrix} -U\Sigma\Sigma^TU^T & 0 \\ 0 & -V\Sigma^T\Sigma V^T \end{bmatrix} \quad (23)$$

Accordingly:

$$\{\mu_i^2\} = \sigma(T_J^2) = (\sigma(T_J))^2 = \sigma(-FF^T) = \{-S_i^2\}. i = 1,2,...,q$$

Theorem 2

Let T_{KSOR} and T_J be given, respectively, by (14) and (13). Then the eigenvalues $\mu_i \in \sigma(T_J)$ and $\beta_i \in \sigma(T_{KSOR})$ are linked by the functional relation Equation 24:

$$(\beta_i + \omega^*\beta_i - 1)^2 = \omega^{*2}\mu_i\beta_i \quad (24)$$

Moreover, the eigenvalues and 2-norm of matrices T_{KSOR} and $\Delta(\omega^*)$ are related as follows Equation 25-29:

$$\sigma(T_{KSOR}) = \sigma(\Delta(\omega^*)) \quad (25)$$

$$\rho(T_{KSOR}) = \rho(\Delta(\omega^*)) = \max_{1\le i\le q} \rho(\Delta_i(\omega^*)) \quad (26)$$

$$\| T_{KSOR}^k \|_2 = \| \Delta^k(\omega^*) \|_2 = \max_{1\le i\le q} = \| \Delta_i^k(\omega^*) \|_2 \quad (27)$$

Where:

$$\Delta(\omega^*) = \mathrm{diag}(\Delta_1(\omega^*),\cdots,\Delta_q(\omega^*),\frac{1}{\omega^*+1}I_{p-q}) \quad (28)$$

$$\Delta_i(\omega^*) = \begin{pmatrix} \frac{1}{1+\omega^*} & \frac{\omega^*}{1+\omega^*}s_i \\ \frac{-\omega^*}{(1+\omega^*)^2}s_i & \frac{1}{1+\omega^*} - \frac{\omega^{*2}}{(1+\omega^*)^2}s_i^2 \end{pmatrix} i = 1,.q \quad (29)$$

where, s_i are the singular values of F.

Proof

By using the singular value decomposition of the matrix F we have $F = U\Sigma V^T$ where U and V orthogonal matrices, then the matrix T_{KSOR} has the form Equation 30:

$$T_{KSOR} = \begin{pmatrix} \frac{1}{1+\omega^*} I_P & \frac{\omega^*}{1+\omega^*} U\Sigma V^T \\ \frac{-\omega^*}{(1+\omega^*)^2} V\Sigma^T U^T & \frac{1}{1+\omega^*} I_q - \frac{\omega^{*2}}{(1+\omega^*)^2} V\Sigma^T\Sigma V^T \end{pmatrix} \quad (30)$$

Let the orthogonal matrices U and V be factored out then T_{KSOR} has the form Equation 31:

$$T_{KSOR} = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \frac{1}{1+\omega^*} I_P & \frac{\omega^*}{1+\omega^*}\Sigma \\ \frac{-\omega^*}{(1+\omega^*)^2}\Sigma^T & \frac{1}{1+\omega^*} I_q - \frac{\omega^{*2}}{(1+\omega^*)^2}\Sigma^T\Sigma \end{bmatrix} \begin{bmatrix} U^T & 0 \\ 0 & V^T \end{bmatrix} \quad (31)$$

Note that (31) reveals the unitarily equivalent matrix Γ_{ω^*} with four block submatrices, each of which is a diagonal sub-matrix where Equation 32:

$$\Gamma_{\omega^*} = \begin{bmatrix} \frac{1}{1+\omega^*} I_P & \frac{\omega^*}{1+\omega^*}\Sigma \\ \frac{-\omega^*}{(1+\omega^*)^2})\Sigma^T & \frac{1}{1+\omega^*} I_q - \frac{\omega^{*2}}{(1+\omega^*)^2}\Sigma^T\Sigma \end{bmatrix} \text{and}$$
$$Q = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}, Q^T = \begin{bmatrix} U^T & 0 \\ 0 & V^T \end{bmatrix} \quad (32)$$

This mean that there is a permutation matrix P which "pulls" the two corner diagonal matrices to the main diagonal, i.e., $P\Gamma_{\omega^*}P^T$ has only 2×2 or 1×1 matrices along its main diagonal. When Γ_{ω^*} of (32) is permuted into the block diagonal form, we obtain Equation 33:

$$\Delta(\omega^*) = P\Gamma_{\omega^*}P^T = \text{diag} \left(\Delta_1(\omega^*), \ldots\ldots, \Delta_q(\omega^*), \frac{1}{\omega^*+1} I_{p-q}\right) \quad (33)$$

where each 2×2 matrix $\Delta_i(\omega^*)$ is given by Equation 34:

$$\Delta_i(\omega^*) = \begin{pmatrix} \frac{1}{1+\omega^*} & \frac{\omega^*}{1+\omega^*} s_i \\ \frac{-\omega^*}{(1+\omega^*)^2} s_i & \frac{1}{1+\omega^*} - \frac{\omega^{*2}}{(1+\omega^*)^2} s_i^2 \end{pmatrix} i = 1, \cdots, q \quad (34)$$

where, s_i are the singular values of F.

We have seen that each member of the ω^* family of KSOR iteration matrix T_{KSOR} is unitarily equivalent to a matrix Δ (ω^*) having only 2×2 or 1×1 matrices on the main diagonal. That is, from (31) and (33) Equation 35:

$$T_{KSOR} = Q\, P^T \Delta(\omega^*) P Q^T \text{ for unitary}(QP^T) \quad (35)$$

Unitary equivalent (35) implies that both the eigenvalues and the 2-norms agree for both (ω^*-families of) matricies T_{KSOR} and $\Delta(\omega^*)$ then we have (25), (26) and (27). From (25) we have Equation 36:

$$\det(\beta_i I_m - T_{KSOR}) = 0 \text{ iff } \det(\beta_i I_m - \Delta(\omega^*)) = 0 \quad (36)$$

From the right-hand determinant above, we see, from (33), (34), that all β_i are constrained by Equation 37:

$$\beta_i = \frac{1}{\omega^*+1}, \text{or} \det(\beta_i I_2 - \Delta_i(\omega^*)) = 0; i = 1, 2, \ldots, q \quad (37)$$

Then:

$$\det(\beta_i I_2 - \Delta_i(\omega^*)) = \begin{pmatrix} \beta_i - \frac{1}{1+\omega^*} & -\frac{\omega^*}{1+\omega^*} s_i \\ \frac{\omega^*}{(1+\omega^*)^2} s_i & \beta_i - \frac{1}{1+\omega^*} + \frac{\omega^{*2}}{(1+\omega^*)^2} s_i^2 \end{pmatrix} = 0$$

$$\det(\beta_i I_2 - \Delta_i(\omega^*)) = \left(\beta_i - \frac{1}{1+\omega^*}\right)^2 + \frac{\omega^{*2}}{(1+\omega^*)^2} S_i^2 \left(\beta_i - \frac{1}{1+\omega^*}\right) + \frac{\omega^{*2}}{(1+\omega^*)^3} S_i^2 = 0$$

$$\left(\beta_i - \frac{1}{1+\omega^*}\right)^2 + \frac{\omega^{*2}}{(1+\omega^*)^2} s_i^2 \beta_i = 0$$

Thus we find:

$$\beta_i = \frac{1}{\omega^*+1}, \text{or } (\beta_i + \omega^*\beta_i - 1)^2 + \omega^{*2} s_i^2 \beta_i = 0 \; i = 1, 2, \ldots q$$

Accordingly, with the help of (15) we can write Equation 38:

$$\beta_i = \frac{1}{\omega^*+1}, \text{or } (\beta_i + \omega^*\beta_i - 1)^2 = \omega^{*2} \mu_i^2 \beta_i \; i = 1, 2, \ldots q \quad (38)$$

Now the left-hand equation, $\beta_i = \frac{1}{\omega^* + 1}$ appears in (33), (34) once for each occurrence of a zero eigenvalue for T_J, but $\beta_i = \frac{1}{\omega^* + 1}$ is a special case of the right-hand side of (38), namely, when μ_i is set to zero. Therefore, (38) is described by the single relation (24).

From the previous theorem we see the ℓ_2-norm of the KSOR iteration matrix is equivalent to the ℓ_2-norm of the $\Delta(\omega^*)$ then, equivalent to the square root of the spectral radius of $(\Delta(\omega^*))^T \Delta(\omega^*)$. Then, the problem of minimizing the ℓ_2-norm of the KSOR iteration matrix is equivalent to the problem of minimizing the square root of the spectral radius of $(\Delta(\omega^*))^T \Delta(\omega^*)$.

Theorem 3

Under the assumptions of the theorem 2, for K = 1 the minimum of the of ℓ_2-norm of the T_{KSOR} is equivalent to Equation 39-41:

$$\delta^2 =: \min_{\omega^* \in R-[-2,0]} \delta_1^{\,2} = \min_{\omega^* \in R-[-2,0]} \max\left\{\frac{1}{2}[T(t) + [T^2(t) - 4C]^{1/2}], \frac{1}{\left(1+\omega^*\right)^2}\right\} \quad (39)$$

Where:

$$T\left(\omega^*, t\right) := \frac{2}{(1+\omega^*)^2} + \frac{\omega^{*4} t}{(1+\omega^*)^4}(1+t) \quad (40)$$

And:

$$C(\omega^*) := \frac{1}{(1+\omega^*)^4} \quad (41)$$

With t is the square of the spectral radius of the Jacobi iteration matrix T_J.

Proof

From the theorem 2 we have Equation 42:

$$\| T_{KSOR} \|_2 = \| \Delta(\omega^*) \|_2 = \max_{1\le i\le q} \| \Delta_i(\omega^*) \|_2 = \max_{1\le i\le q} \{\rho^{1/2}(\Delta_i^{\,T}(\omega^*)\Delta_i(\omega^*))\} \quad (42)$$

Now we go to calculate Equation 43 and 44:

$$\Delta_i^{\,T}\left(\omega^*\right)\Delta_i\left(\omega^*\right) = \begin{pmatrix} \frac{1}{1+\omega^*} & \frac{-\omega^*}{(1+\omega^*)^2} s_i \\ \frac{\omega^*}{1+\omega^*} s_i & \frac{1}{1+\omega^*} - \frac{\omega^{*2}}{(1+\omega^*)^2} s_i^{\,2} \end{pmatrix} \begin{pmatrix} \frac{1}{1+\omega^*} & \frac{\omega^*}{1+\omega^*} s_i \\ \frac{-\omega^*}{(1+\omega^*)^2} s_i & \frac{1}{1+\omega^*} - \frac{\omega^{*2}}{(1+\omega^*)^2} s_i^{\,2} \end{pmatrix} \quad (43)$$

$$\Delta_i^T(\omega^*)\Delta_i(\omega^*) = \begin{bmatrix} \frac{1}{\left(1+\omega^*\right)^2}\left(1 + \frac{\omega^{*^2} s_i^{\,2}}{\left(1+\omega^*\right)^2}\right) \\ \frac{\omega^{*2} s_i}{\left(1+\omega^*\right)^3}\left(1 + \frac{\omega^* s_i^{\,2}}{(1+\omega^*)}\right) \\ \frac{\omega^{*^2} s_i^{\,2}}{\left(1+\omega^*\right)^3}\left(1 + \frac{\omega^* s_i^{\,2}}{(1+\omega^*)}\right) \\ \frac{\omega^{*^2} s^2}{\left(1+\omega^*\right)^3}\left(\omega^* - 1 + \frac{\omega^{*^2} s_i^{\,2}}{1+\omega^*}\right) + \frac{1}{\left(1+\omega^*\right)^2} \end{bmatrix} \quad (44)$$

It is easy to see Equation 45:

$$\det(\Delta_i^{\,T}(\omega^*)\Delta_i(\omega^*) - \beta I_2) = \beta^2 - \left(\frac{2}{(1+\omega^*)^2} + \frac{\omega^{*4} S_i^2}{(1+\omega^*)^4}(1+S_i^{\,2})\right)\beta + \frac{1}{(1+\omega^*)^4} \quad (45)$$

Set $s_i^2 = t_i$ and define Equation 46 and 47:

$$c(\omega^*) := \frac{1}{(1+\omega^*)^4} \quad (46)$$

$$T\left(\omega^*, t\right) := \frac{2}{(1+\omega^*)^2} + \frac{\omega^{*4} t}{(1+\omega^*)^4}(1+t) \quad (47)$$

Therefore Equation 48:

$$\det(\Delta_i^{\,T}(\omega^*)\Delta_i(\omega^*)) - \beta I_2) = \beta^2 - T(t_i)\beta + C \quad (48)$$

Solving this quadratic equation, we find that Equation 49:

$$\beta = \frac{1}{2}\{T(t_i) \pm [T^2(t_i) - 4C]^{1/2}\} \quad (49)$$

Note that, for any $t \ge 0$, $\omega^* \in R/[-2,0]$, we have Equation 50:

$$T(t)\rangle 0 \quad T^2(t)-4C\geq 0 \tag{50}$$

Note that: The eigenvalues of the matrix $\Delta_i^T(\omega^*)\Delta_i(\omega^*)$ are nonnegative numbers and form the roots of the char-acteristic Equation 51 and 52:

$$\beta^2 - T(t_i)\beta + C = 0 \quad i = 1,2,....q \tag{51}$$

And:

$$\beta - \frac{1}{(1+\omega^*)^2} = 0 \tag{52}$$

The largest of the two roots of (51) is given by:

$$L_i := L_i(\omega^*) \equiv \frac{1}{2}\{T(\omega^*,t_i) + [T^2(\omega^*,t_i) - 4C(\omega^*)]^{1/2}\} i = 1,2\ldots q$$

The maximum value of each L_i is obtained for the maximum value of the corresponding T (ω^*, t_i).

Note that:

$$\frac{dT(t)}{dt} = \frac{\omega^{*4}}{(1+\omega^*)^4}(2t+1) \geq 0, \text{for any } t \geq 0$$

Now for any $t \rangle 0$, T(t) is a strictly increasing function of t. Likewise, L_i is strictly increasing function of t_i, set Equation 53 and 54:

$$L := L(\omega^*) = \max_{i=1,2,\cdots,q} L_i \tag{53}$$

Then:

$$L := L(\omega^*) = \frac{1}{2}\left\{T(\omega^*,t) + \left(T^2(\omega^*,t) - 4C(\omega^*)\right)^{1/2}\right\} \tag{54}$$

With $t = \rho^2(T_J)$.

The spectral radius of the matrix $\Delta_i^T(\omega^*)\ \Delta_i(\omega^*)$ for any given i, is the quantity L_i, then from (53) the spectral radius of the matrix $\Delta^T(\omega^*)\ \Delta(\omega^*)$ is L.

Theorem 4

The value of ω^*, which has minimum in (39), is the unique real positive root in $(0,\infty)$ of the Equation 55:

$$f(\omega^*) = (t+t^2)\omega^{*2} - \omega^* + 1 = 0 \tag{55}$$

Proof

From (40) we see Equation 56 and 57:

$$T(\omega^*,t) := \frac{2}{(1+\omega^*)^2} + \frac{\omega^{*4}t}{(1+\omega^*)^4} \tag{56}$$

$(1+t)\rangle 0$ for any $\omega^* \in R/[-2,0]$

And:

$$\frac{dT(\omega^*,t)}{d\omega^*} := \frac{-4}{(1+\omega^*)^3} + \frac{4\omega^{*3}t}{(1+\omega^*)^5}(1+t) \tag{57}$$

Then we find Equation 58:

$$\frac{dT(\omega^*,t)}{d\omega^*}\rangle 0 \text{ for any } \omega^*\langle -2 \tag{58}$$

The function $T(\omega^*, t)$ increases strictly in the interval $(-\infty, -2)$.

Differentiating $L(\omega^*, t)$ defined in (54) with respect to ω^*, and using (40) and (41) we find Equation 59:

$$\frac{dL}{d\omega^*} = \frac{1}{2}\left\{\frac{dT}{d\omega^*} + \frac{T\left(\frac{dT}{d\omega^*}\right) + \frac{8}{(1+\omega^*)^5}}{\left[T^2 - \frac{4}{(1+\omega^*)^4}\right]^{1/2}}\right\} \tag{59}$$

It is clear that Equation 60:

$$\frac{dL}{d\omega^*}\rangle 0 \text{ for any } \omega^* \in (-\infty,-2) \tag{60}$$

So that, the function $L(\omega^*, t)$ increases strictly in the interval $(-\infty,-2)$. We will take limit of the function $\frac{dL}{d\omega^*}$ as $\omega^*\to\infty$, we obtain Equation 61:

$$\lim_{\omega^*\to 0}\frac{dL}{d\omega^*} = \frac{1}{2}\left\{\lim_{\omega^*\to 0}\frac{dT}{d\omega^*} + \lim_{\omega^*\to 0}\frac{T\left(\frac{dT}{d\omega^*}\right) + \frac{8}{(1+\omega^*)^5}}{\left[T^2 - \frac{4}{(1+\omega^*)^4}\right]^{1/2}}\right\} \tag{61}$$

Then when, $\omega^* \to \infty$ we have $\frac{dL}{d\omega^*} \to 0^+$, now we take limit as $\omega^* \to \infty$ and obtain Equation 62:

$$\lim_{\omega^*\to 0}\frac{dL}{d\omega^*}=\frac{1}{2}\left\{\lim_{\omega^*\to 0}\frac{dT}{d\omega^*}+\lim_{\omega^*\to 0}\frac{T\left(\frac{dT}{d\omega^*}\right)+\frac{8}{(1+\omega^*)^5}}{\left[T^2-\frac{4}{(1+\omega^*)^4}\right]^{1/2}}\right\} \quad (62)$$

Set Equation 63 and 64:

$$W=T\left(\frac{dT}{d\omega^*}\right)+\frac{8}{(1+\omega^*)^5} \quad (63)$$

And:

$$V=T^2-\frac{4}{(1+\omega^*)^4} \quad (64)$$

Now simplify (63) and (64) as:

$$\frac{w}{V}=\frac{4\omega^*\sqrt{t^2+t}\left(2-\omega^*+4\left(t^2+t\right)\frac{\omega^{*4}}{\left(1+\omega^*\right)^2}\right)}{(1+\omega^*)^4\left[\frac{\omega^{*4}}{\left(1+\omega^*\right)^2}\left(t^2+t\right)+4\right]}\to 0 \text{ as } \omega^*\to 0$$

Then we have Equation 65:

$$\lim_{\omega^*\to 0}\frac{dL}{d\omega^*}=-2\langle 0 \quad (65)$$

Therefore, from (61) and (65) L(ω^*, t) has a odd number of local minimum points in (0, ∞).

For any fixed t ∈(0, 1), the global minimum point of L(ω^*, t) is a point in (0, ∞) at which $\frac{dL}{d\omega^*}$ vanishes. Setting $\frac{dL}{d\omega^*}=0$ then:

$$\frac{dT}{d\omega^*}\left[T^2-\frac{4}{\left(1+\omega^*\right)^4}\right]^{1/2}=-T\frac{dT}{d\omega^*}-\frac{8}{(1+\omega^*)^5}$$

Then:

$$\left(\frac{dT}{d\omega^*}\sqrt{T^2-\frac{4}{\left(1+\omega^*\right)^4}}\right)^2=\left(-T\frac{dT}{d\omega^*}-\frac{8}{(1+\omega^*)^5}\right)^2$$

That is:

$$\left(\frac{dT}{d\omega^*}\right)^2\left(T^2-\frac{4}{\left(1+\omega^*\right)^4}\right)=\left(T\frac{dT}{d\omega^*}\right)^2+T\frac{dT}{d\omega^*}\frac{16}{(1+\omega^*)^5}+\frac{64}{(1+\omega^*)^{10}}$$

Eliminating (T dT/dω^*)^2 and dividing through -4, we obtain:

$$\frac{1}{\left(1+\omega^*\right)^4}\left(\frac{dT}{d\omega^*}\right)^2+T\frac{dT}{d\omega^*}\frac{4}{(1+\omega^*)^5}+\frac{16}{(1+\omega^*)^{10}}=0$$

It now follows that:

$$\left(\frac{dT}{d\omega^*}\right)^2+T\frac{dT}{d\omega^*}\frac{4}{1+\omega^*}+\frac{16}{(1+\omega^*)^6}=0$$

Substituting (56) for T and (57) for dT/dω^*, we obtain Equation 66:

$$f\left(\omega^*\right)=\left(t^2+t\right)\omega^{*^2}-\omega^*-1 \quad (66)$$

Then, we have Equation 67 and 68:

$$r_1\left(\omega^*\right)=\frac{1+\sqrt{1+4\left(t+t^2\right)}}{2\left(t+t^2\right)}=\frac{1}{t} \quad (67)$$

$$r_2\left(\omega^*\right)=\frac{1-\sqrt{1+4\left(t^2+t\right)}}{2\left(t+t^2\right)}=\frac{-1}{t+1}\langle 0 \quad (68)$$

Therefore, f (ω^*) has a unique zero $r_1(\omega^*)$ in that interval. So the $r_1(\omega^*)$ is a unique real positive root in (0, ∞) of the equation (66), from that and (61), we notes that Equation 69:

$$L\left(r_1\left(\omega^*\right),t\right)\rangle \lim_{\omega^*\to\pm\infty} L\left(\omega^*,t\right) \quad (69)$$

So that $r_1(\omega^*)$ is a unique real positive root in (0, ∞) of the equation which has the minimum of L(ω^*).

Example

Consider a system with:

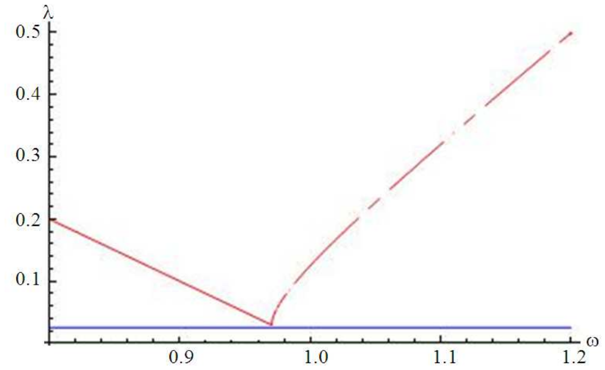

Fig. 1. The behviour of the spectral radius of the T_{SOR} as a function of ω

$$A = \begin{bmatrix} 1.0 & 0 & 0.25 & -0.25 \\ 0 & 1.0 & 0.25 & 0.25 \\ -0.25 & -0.25 & 1.0 & 0 \\ 0.25 & -0.25 & 0 & 1.0 \end{bmatrix}, b = \begin{bmatrix} 1.0 \\ 1.5 \\ 0.5 \\ 1.0 \end{bmatrix}$$

For simplicity we adapted the right hand side b as it was in Young (2003) and Youssef (2012) so that the exact solution is $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$.

It is well known that, for this system we have.

The eigenvalues of the Jacobi iteration matrix T_J are the roots of the equation:

$$\mu^4 + 0.25\mu^2 + 0.015625 = 0$$

The roots are:

$$\mu_1 = \mu_2 = 0.353553I, \mu_3 = \mu_4 = -0.353553I$$

For the matrix, $F = \begin{bmatrix} -0.25 & 0.25 \\ -0.25 & -0.25 \end{bmatrix}$ the Singular values are $s_1 = s_2 = 0.353553$.

It is clear that the Jacobi iteration matrix T_J is a skew symmetric, accordingly their eigenvalues are pure imaginary complex numbers, and satisfies $\mu_i^2 = s_i^2$.

3. RESULTS

- We used the SVD in proving the eigenvalue functional relation for the KSOR operator
- The minimization of the ℓ_2-norm is used as a good estimation for determining the optimum value of the relaxation parameter in the KSOR method as well as in the SOR method

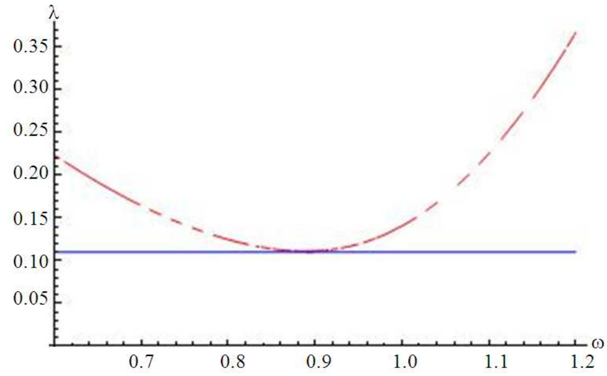

Fig. 2. The behaviour of the ℓ_2-norm of T_{SOR} as a function of ω

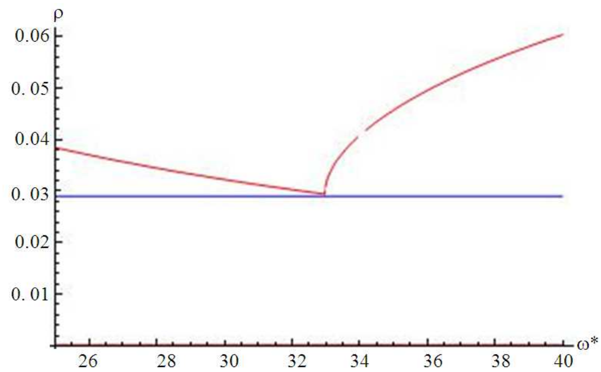

Fig. 3. The behviour of the spectral radius of the T_{KSOR} as a function of ω^*

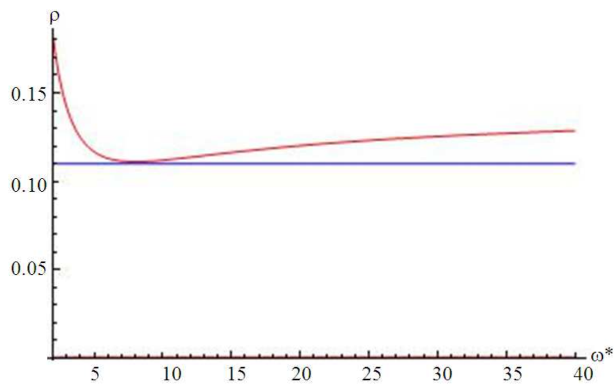

Fig. 4. The behaviour of the ℓ_2-norm of T_{SOR} as a function of ω^*

- From **Fig. 1 and 2** we see that the calculated results agree with the theoretical results of Milleo *et al*. (2006)

- From **Fig. 3 and 4** we see that the calculated results agree with our theoretical results
- Numerical example illustrating and confirming the theoretical relations is considered

4. DISCUSION

Young (2003), considered the problem why convergence of the SOR method with the optimum ω_{opt} in the sense of minimizing the spectral radius of the iteration matrix is some what slower than what might expected, the spectral radius is only an asymptotic measure of the rate of convergence of a linear iterative method. In his treatment Young (2003), established a relation between the eigenvalues of certain matrices related to A (the SOR iteration matrix, T_{SOR}) and those of certain block 2×2 matrices.

Golub and Pillis (1990) raised the question of determining, for each $k \geq 1$, a relaxation parameter $\omega \in (0, 2)$ which minimizes the Euclidean norm of the k^{th} power of the SOR iteration matrix, associated with a real symmetric positive definite matrix with "property A".

Hadjidimos and Neumann (1998), used the reduction of the SOR operator introduced by Golub and Pillis (1990), with the help of the SVD of the associated block Jacobi iteration matrix to obtain the minimizing relaxation parameter for the case k = 1. Yin and Yuan (2002), used the SVD to re-derive the eigenvalue functional relations for block skew symmetric matrices for the AOR method. Milleo *et al*. (2006), considered systems with block skew symmetric Jacobi iteration matrix and used the SVD in studying the behavior of the SOR operator from the ℓ_2-norm point of view they determined theoretically the minimizing relaxation parameter of the ℓ_2-norm. Youssef (2012), defined the KSOR operator, we used the SVD in re-prove the functional eigenvalue relation for the KSOR operator and the corresponding unitary block 2×2 matrix $\Delta(\omega^*)$. we employed the same argument as in Yin and Yuan (2002), also in Milleo *et al*. (2006) for systems with block skew symmetric Jacobi iteration matrix and used the SVD in studying the behavior of the ℓ_2-norm of the KSOR operator. We determined theoretically the minimizing relaxation parameter of the ℓ_2-norm for the KSOR operator. We confirmed our theoretical results by a numerical example. We will continue this study in a subsequent work in which we will consider a generalizations of the KSOR operator.

5. CONCLUSION

We used the same argument defined by Golub and Pillis (1990), used by Yin and Yuan (2002) also by Milleo *et al*. (2006), we proved that the KSOR iteration matrix T_{KSOR} is unitary equivalent to a matrix $\Delta(\omega^*)$ having only 2×2 or 1×1 matrices on the diagonal. We minimize the ℓ_2-norm of the KSOR operator for matrices whose Jacobi iteration matrix is skew symmetric. By our results, the optimal value of the relaxation parameter is:

$$\omega^*_{opt} = \frac{1}{(\rho(T_J))^2}.$$

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