

Half-bounded Numerical Solution of Singular Integral Equations with Cauchy Kernel

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Abstract: Problem statement: In this study, a numerical solution for singular integral equations of the first kind with Cauchy kernel over the finite segment $[-1,1]$ is presented. The numerical solution is bounded at $x=1$ and unbounded at $x=-1$. **Approach:** The numerical solution is derived by approximating the unknown density function using the weighted Chebyshev polynomials of the fourth kind. **Results:** The force function is approximated by using the Chebyshev polynomials of the third kind. **Conclusion:** The exactness of the numerical solution is shown for characteristic equation when the force function is a cubic.

Key words: Singular integral equations, Cauchy kernel, chebyshev orthogonal polynomials

INTRODUCTION

Consider the Cauchy type singular integral equations of the first kind:

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 K(x,t)\varphi(t) dt = f(x) \quad (1)$$

where $K(x,t)$ and $f(x)$ are assumed to be real-valued functions belong to the class of Hölder on the sets $[-1,1] \times [-1,1]$ and $[-1,1]$, respectively. $\varphi(x)$ is unknown function to be determined. The theories and applications of singular integral equations are described in many references as well as in Gakhov (1990), Ladopoulos (2000), Lifanov (1996) and Nandhakumar *et al.* (2009).

The characteristic singular integral equation is of the form (Lifanov (1996):

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt = f(x), \quad -1 < x < 1 \quad (2)$$

It is well known that the analytical solution of characteristic equation (2) which is bounded at the endpoint $x=1$ is given by the following formula:

$$\varphi(x) = -\frac{1}{\pi^2} \sqrt{\frac{1-x}{1+x}} \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{f(t)}{t-x} dt \quad (3)$$

By solving equation (1) with respect to its characteristic part, we will find that it is equivalent to the Fredholm equation of the second kind:

$$\left. \begin{aligned} \varphi(t) + \int_{-1}^1 N(t,\tau)\varphi(\tau) d\tau &= F(t), \\ N(t,\tau) &= -\frac{1}{\pi^2} \sqrt{\frac{1-t}{1+t}} \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} \frac{K(x,\tau)}{x-t} dx, \\ F(t) &= -\frac{1}{\pi^2} \sqrt{\frac{1-t}{1+t}} \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} \frac{f(x)}{x-t} dx, \end{aligned} \right\} \quad (4)$$

in the sense of obtaining the solution, which one can apply the Fredholm's theorems.

Mahiub *et al.* (2009) presented a numerical solution of equation (1) which is unbounded at both the endpoints $x = \pm 1$. They used Chebyshev polynomials of the first kind with the corresponding weight function to approximate the unknown density function. They proved that the numerical solution of characteristic equation is identical to the exact solution when the force function is a cubic.

Abdulkawi *et al.* (2009a) discussed a numerical solution of equation (1) which is bounded at the both endpoints $x = \pm 1$. They used Chebyshev polynomials of the second kind with the corresponding weight function to approximate the density function. They showed that the numerical solution of characteristic equation is identical with the exact solution when the force function is a cubic.

Abdulkawi *et al.* (2009b) presented an approximate solution of equation (1) which is bounded at the endpoint $x = -1$, but unbounded at the endpoint $x = 1$. They used Chebyshev polynomials of the third kind with the corresponding weight function to approximate the unknown density function. They proved that the

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Exactness of the numerical solution:

Proposition 1. If $f(x)$ in equation (2) is a cubic function, then the approximate solution (9) of characteristic equation (2) is exact.

Proof: Let $f(x)$ in equation (2) be a cubic function, i.e:

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 \tag{17}$$

Substituting (17) into (2), we have:

$$\int_{-1}^1 \frac{\phi(t)}{t-x} dt = c_0 + c_1x + c_2x^2 + c_3x^3, \quad -1 < x < 1 \tag{18}$$

Substitution (17) into (8) yields:

$$\hat{f}_k = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} (c_0 + c_1t + c_2t^2 + c_3t^3) V_k(t) dt \tag{19}$$

The Chebyshev recurrence relations of the third and fourth kinds, respectively, are:

$$\left. \begin{aligned} V_0(x) &= 1, \quad V_1(x) = 2x - 1, \\ V_n(x) &= 2xV_{n-1}(x) - V_{n-2}(x), \quad n \geq 2, \end{aligned} \right\} \tag{20}$$

and:

$$\left. \begin{aligned} W_0(x) &= 1, \quad W_1(x) = 2x + 1, \\ W_n(x) &= 2xW_{n-1}(x) - W_{n-2}(x), \quad n \geq 2. \end{aligned} \right\} \tag{21}$$

With help of (20) and (21), we have:

$$\left. \begin{aligned} t^3 &= \frac{1}{8} [V_3(t) + V_2(t) + 3(V_1(t) + V_0(t))] \\ &= \frac{1}{8} [W_3(t) - W_2(t) + 3(W_1(t) - W_0(t))], \\ t^2 &= \frac{1}{4} [V_2(t) + V_1(t) + 2V_0(t)] \\ &= \frac{1}{4} [W_2(t) - W_1(t) + 2W_0(t)], \\ t &= \frac{1}{2} [V_1(t) + V_0(t)] \\ &= \frac{1}{2} [W_1(t) - W_0(t)]. \end{aligned} \right\} \tag{22}$$

It is known that the Chebyshev polynomials of the third and fourth kinds are orthogonal with respect to the weight functions ω_1 and ω_2 , respectively, i.e.:

$$\int_{-1}^1 \sqrt{\frac{1+t}{1-t}} V_m(t) V_n(t) dt = \begin{cases} 0, & n \neq m, \\ \pi, & n = m, \end{cases} \tag{23}$$

and:

$$\int_{-1}^1 \sqrt{\frac{1-t}{1+t}} W_m(t) W_n(t) dt = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases} \tag{24}$$

Taking into account (22) and using the orthogonal property (23) into (19), we obtain:

$$\left. \begin{aligned} \hat{f}_0 &= c_0 + \frac{c_1 + c_2}{2} + \frac{3c_3}{8}, \\ \hat{f}_1 &= \frac{2c_1 + c_2}{4} + \frac{3c_3}{8}, \\ \hat{f}_2 &= \frac{2c_2 + c_3}{8}, \\ \hat{f}_3 &= \frac{c_3}{8}. \end{aligned} \right\} \tag{25}$$

From the system (16) where $K(x, t) = 0$ yields:

$$b_j = -\frac{1}{\pi} \hat{f}_j, \quad j = 0, \dots, n \tag{26}$$

Substituting (26) into (9) for $n = 3$, we get:

$$\varphi_n(x) = -\frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} \times \left[\hat{f}_0 + \hat{f}_1 W_1(x) + \hat{f}_2 W_2(x) + \hat{f}_3 W_3(x) \right] \tag{27}$$

Thus the approximate solution of equation (18) is:

$$\left. \begin{aligned} \phi_n(x) &= -\frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} p(x), \\ p(x) &= c_0 + c_1 + \frac{1}{2}(c_2 + c_3) \\ &\quad + (c_1 + c_2 + \frac{1}{2}c_3)x + (c_2 + c_3)x^2 + c_3x^3. \end{aligned} \right\} \tag{28}$$

In order to obtain the exact solution of equation (18), we substitute (17) into (3) which gives:

$$\varphi(x) = -\frac{1}{\pi^2} \sqrt{\frac{1-x}{1+x}} \times \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{c_0 + c_1t + c_2t^2 + c_3t^3}{t-x} dt \tag{29}$$

It is easy to see that:

$$\left. \begin{aligned} \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{1}{t-x} dt &= \pi, \\ \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{t}{t-x} dt &= \pi(1+x), \\ \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{t^2}{t-x} dt &= \pi(0.5 + x + x^2), \\ \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} \frac{t^3}{t-x} dt &= \pi(0.5 + 0.5x + x^2 + x^3). \end{aligned} \right\} \tag{30}$$

From (29)-(30) we obtain the exact solution of equation (18) which is identical to the approximate solution given by (28). The proof is complete.

RESULTS AND DISCUSSION

In this section, we discuss a particular example to show the accuracy of our numerical solution.

Let us consider the integral equation (1) with degenerate kernel $k(x, t) = x^5 t^5$ and polynomial function $f(x) = x^5 + x^3 + x$, i.e.:

$$\int_{-1}^1 \frac{\varphi(t)}{t-x} dt + \int_{-1}^1 x^5 t^5 \varphi(t) dt = x^5 + x^3 + x \tag{31}$$

and we seek the solution of this equation, which is bounded at the endpoint $x = 1$ and unbounded at the endpoint $x = -1$.

Due to (8), we have:

$$\hat{f}_k = \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1+t}{1-t}} (t^5 + t^3 + t) V_k(t) dt \tag{32}$$

It is easy to see that:

$$\left. \begin{aligned} t^5 &= \frac{1}{32} [V_5(t) + V_4(t) + 5(V_3(t) + V_2(t)) \\ &\quad + 10(V_1(t) + V_0(t))] \\ &= \frac{1}{32} [W_5(t) - W_4(t) + 5(W_3(t) - W_2(t)) \\ &\quad + 10(W_1(t) - V_0(t))]. \end{aligned} \right\} \tag{33}$$

From (22), (32) and (33) we obtain:

$$\hat{f}_0 = \hat{f}_1 = \frac{19}{16}, \hat{f}_2 = \hat{f}_3 = \frac{9}{32}, \hat{f}_4 = \hat{f}_5 = \frac{1}{32} \tag{34}$$

From (14) where $K(x, t) = x^5 t^5$, we have:

$$\begin{aligned} \lambda_{jk} &= \frac{1}{\pi} \int_{-1}^1 \sqrt{\frac{1+x}{1-x}} x^5 \int_{-1}^1 \sqrt{\frac{1-t}{1+t}} t^5 W_j(t) V_k(x) dt dx \\ &= \frac{1}{\pi} \left[\int_{-1}^1 \sqrt{\frac{1+x}{1-x}} x^5 V_k(x) dx \right] \\ &\quad \times \left[\int_{-1}^1 \sqrt{\frac{1-t}{1+t}} t^5 W_j(t) dt \right] \end{aligned} \tag{35}$$

Due to (23), (24) and (33), yields:

$$\left. \begin{aligned} \lambda_{00} = \lambda_{01} &= -\frac{100\pi}{(32)^2}, \lambda_{02} = \lambda_{03} = -\frac{50\pi}{(32)^2}, \\ \lambda_{04} = \lambda_{05} &= -\frac{10\pi}{(32)^2}, \lambda_{10} = \lambda_{11} = \frac{100\pi}{(32)^2}, \\ \lambda_{12} = \lambda_{13} &= \frac{50\pi}{(32)^2}, \lambda_{14} = \lambda_{15} = \frac{10\pi}{(32)^2}, \\ \lambda_{20} = \lambda_{21} &= -\frac{50\pi}{(32)^2}, \lambda_{22} = \lambda_{23} = -\frac{25\pi}{(32)^2}, \\ \lambda_{24} = \lambda_{25} &= -\frac{5\pi}{(32)^2}, \lambda_{30} = \lambda_{31} = \frac{50\pi}{(32)^2}, \\ \lambda_{32} = \lambda_{33} &= \frac{25\pi}{(32)^2}, \lambda_{34} = \lambda_{35} = \frac{5\pi}{(32)^2}, \\ \lambda_{40} = \lambda_{41} &= -\frac{10\pi}{(32)^2}, \lambda_{42} = \lambda_{43} = -\frac{5\pi}{(32)^2}, \\ \lambda_{44} = \lambda_{4,5} &= -\frac{\pi}{(32)^2}, \lambda_{50} = \lambda_{51} = \frac{10\pi}{(32)^2}, \\ \lambda_{52} = \lambda_{53} &= \frac{5\pi}{(32)^2}, \lambda_{54} = \lambda_{55} = \frac{\pi}{(32)^2}. \end{aligned} \right\} \tag{36}$$

Then the unknown coefficients $b_j, j=0, \dots, 5$ are obtained by solving the following system of linear equations:

$$\left. \begin{aligned} b_0 + \frac{100}{(32)^2} b_0 - \frac{100}{(32)^2} b_1 + \frac{50}{(32)^2} b_2 \\ - \frac{50}{(32)^2} b_3 + \frac{10}{(32)^2} b_4 - \frac{10}{(32)^2} b_5 &= \frac{-19}{16\pi}, \\ b_1 + \frac{100}{(32)^2} b_0 - \frac{100}{(32)^2} b_1 + \frac{50}{(32)^2} b_2 \\ - \frac{50}{(32)^2} b_3 + \frac{10}{(32)^2} b_4 - \frac{10}{(32)^2} b_5 &= \frac{-19}{16\pi}, \\ b_2 + \frac{50}{(32)^2} b_0 - \frac{50}{(32)^2} b_1 + \frac{25}{(32)^2} b_2 \\ - \frac{25}{(32)^2} b_3 + \frac{5}{(32)^2} b_4 - \frac{5}{(32)^2} b_5 &= \frac{-9}{32\pi}, \\ b_3 + \frac{50}{(32)^2} b_0 - \frac{50}{(32)^2} b_1 + \frac{25}{(32)^2} b_2 \\ - \frac{25}{(32)^2} b_3 + \frac{5}{(32)^2} b_4 - \frac{5}{(32)^2} b_5 &= \frac{-9}{32\pi}, \\ b_4 + \frac{10}{(32)^2} b_0 - \frac{10}{(32)^2} b_1 + \frac{5}{(32)^2} b_2 \\ - \frac{5}{(32)^2} b_3 + \frac{1}{(32)^2} b_4 - \frac{1}{(32)^2} b_5 &= \frac{-1}{32\pi}, \\ b_5 + \frac{10}{(32)^2} b_0 - \frac{10}{(32)^2} b_1 + \frac{5}{(32)^2} b_2 \\ - \frac{5}{(32)^2} b_3 + \frac{1}{(32)^2} b_4 - \frac{1}{(32)^2} b_5 &= \frac{-1}{32\pi}, \end{aligned} \right\} \tag{37}$$

which are:

$$\left. \begin{aligned} b_0 = b_1 = -\frac{19}{16\pi}, b_2 = b_3 = -\frac{9}{32\pi}, \\ b_4 = b_5 = -\frac{1}{32\pi}. \end{aligned} \right\} \quad (38)$$

Substituting the values of the coefficients $\{b_j\}_0^5$ into (9) yields the approximate solution of equation (31), which is:

$$\varphi_n(x) = -\frac{1}{\pi} \sqrt{\frac{1-x}{1+x}} \times \left[x^5 + x^4 + \frac{3}{2}(x^3 + x^2) + \frac{15}{8}(x+1) \right] \quad (39)$$

It is not difficult to see that the approximate solution (39) is exact.

CONCLUSION

The Chebyshev orthogonal polynomials of the third and fourth kinds are used to approximate the known force function and unknown density function, respectively. Proposition 1 shows the exactness of the numerical solution for characteristic equation when the force function is a cubic. Particular results show that our numerical solution does not only give the exact solution for characteristic equation but also for other Cauchy singular integral equations.

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