Journal of Mathematics and Statistics 7 (1): 78-80, 2011
ISSN 1549-3644

Remark on Bi-Ideals and Quasi-Ideals of Variants of Regular Rings

[1]Samruam Baupradist and [2,3]Ronnason Chinram
[1]Department of Mathematics, Chulalongkorn University, Bangkok 10330, Thailand
[2]Department of Mathematics and Statistics,
Prince of Songkla University, Hat Yai, Songkhla 90110, Thailand
[3]Centre of Excellence in Mathematics, CHE, Si Ayuthaya Road,
Bangkok 10400, Thailand

Abstract: Problem statement: Every quasi-ideal of a ring is a bi-ideal. In general, a bi-ideal of a ring need not be a quasi-ideal. Every bi-ideal of regular rings is a quasi-ideal, so bi-ideals and quasi-ideals of regular rings coincide. It is known that variants of a regular ring need not be regular. The aim of this study is to study bi-ideals and quasi-ideals of variants of regular rings. **Approach:** The technique of the proof of main theorem use the properties of regular rings and bi-ideals. **Results:** It shows that every bi-ideal of variants of regular rings is a quasi-ideal. **Conclusion:** Although the variant of regular rings need not be regular but every bi-ideal of variants of regular rings is a quasi-ideal.

Key words: Bi-ideals, quasi-ideals, variants, regular rings, BQ-rings

INTRODUCTION

The notion of quasi-ideals in rings was introduced by (Steinfeld, 1953) while the notion of bi-ideals in rings was introduced much later. It was actually introduced (Lajos and Sza'sz, 1971).

For nonempty subsets A, B of a ring R, AB denotes the set of all finite sums of the form $\sum a_i b_i, a_i \in A, b_i \in B$. A subring Q of a ring R is called a quasi-ideal of R if $RQ \cap QR \subseteq Q$ and a bi-ideal of R is a subring B of R such that $BRB \subseteq B$. Every quasi-ideal of R is a bi-ideal. In general, bi-ideals of rings need not be quasi-ideals. See the following example. Consider the ring $(SU_4(\mathbb{R}), +, \cdot)$ of all strictly upper triangular 4×4 matrices over the field $\mathbb{R}$ of real numbers under the usual addition and multiplication of matrices.

$$\text{Let } B = \left\{ \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| x \in \mathbb{R} \right\}.$$

Then B is a zero subring of $(SU_4(\mathbb{R}), +, \cdot)$. Moreover, $BSU_4(\mathbb{R})B = \{0\}$. Thus B isa bi-ideal of $(SU_4(\mathbb{R}), +, \cdot)$.

But

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\in (SU_4(\mathbb{R})B \cap BSU_4(\mathbb{R})) \setminus B.$$

So B is not a quasi-ideal of $(SU_4(\mathbb{R}), +, \cdot)$.

MATERIALS AND METHODS

An element a of a ring R is called regular if there exists x in S such that a = axa. A ring R is called regular if every element in R is regular. The following known result shows a sufficient condition for a bi-ideal of a ring to be a quasi-ideal.

Theorem 1: If B is a bi-ideal of a ring R such that every element of B is regular in R, then B is a quasi-ideal of R. In particular, if R is a regular ring, then every bi-ideal of R is a quasi-ideal.

Let R be a ring and $a \in R$. A new product o defined on R by x o y = xay for all x, $y \in R$. Then (R, +, o) is a

Corresponding Author: Samruam Baupradist, Department of Mathematics, Chulalongkorn University, Bangkok 10330, Thailand

ring. We usually write (R, +, a) rather that (R, +, o) to make the element a explicit. The ring (R, +, a) is called a variant of R with respect to a. It is well-known that the variant of regular rings need not be regular ring (see (Kemprasit, 2002) and (Chinram, 2009).

Our aim is to prove that every bi-ideal of variants of regular rings is a quasi-ideal. In fact, the technique of the proof of Theorem 1 is helpful for our work. However, our proof is more complicated.

RESULTS

The following theorem is our main result.

Theorem 2: Let R be a regular ring and $a\in R$. Then every bi-ideal of the ring (R, +, a) is a quasi-ideal.

Proof: Let B be a bi-ideal of a ring (R, +, a). Then $BaRaB \subseteq B$. To show that $RaB \cap BaR \subseteq B$, let x be an element of $RaB \cap BaR$.

Then:

$$x \in RaB \tag{1}$$

and

$$x = b_{11}ar_1 + b_{12}ar_2 + \ldots + b_{1n}ar_n \tag{2}$$

for some $b_{11}, b_{12} \ldots, b_{1n} \in B$ and $r_1, r_2, \ldots, r_n \in R$.

Since each $b_{1i}a \in R$ and $(R,+,\cdot)$ is a regular ring, there exists $s_{1i} \in R$ such that $b_{1i}a = b_{1i}as_{1i}b_{1i}a$. By (2), we have:

$$x = b_{11}as_{11}b_{11}ar_{11} + b_{12}as_{12}b_{12}ar_2 + \ldots + b_{1n}as_{1n}b_{1n}ar_n \tag{3}$$

and

$$\begin{aligned} b_{11}as_{11}b_{11}ar_1 &= b_{11}as_{11}\left(x - b_{12}ar_2 - \ldots - b_{1n}ar_n\right) \\ &= b_{11}as_{11}x - b_{11}as_{11}b_{12}ar_2 - \ldots - b_{11}as_{11}b_{1n}ar_n. \end{aligned} \tag{4}$$

It then follows from (3) and (4) that:

$$\begin{aligned} x &= b_{11}as_{11}x + \left(b_{12}as_{12}b_{12} - b_{11}as_{11}b_{12}\right)ar_2 \\ &+ \ldots + \left(b_{1n}as_{1n}b_{1n} - b_{11}as_{11}b_{1n}\right)ar_n. \end{aligned}$$

But from (1) and (2):

$$b_{11}as_{11}x \in Bas_{11}RaB \subseteq BaRaB$$

and for $i \in \{2, 3, \ldots, n\}$,

$$b_{1i}as_{1i}b_{1i} - b_1as_{11}b_{1i} \in Bas_{1i}B - Bas_{11}B \subseteq BaR.$$

So:

$$x = b_1 + b_{22}ar_2 + \ldots + b_{2n}ar_n \tag{5}$$

for some $b_1 \in BaRaB$ and $b_{22}, \ldots, b_{2n} \in BaR$.

Since for $i \in \{2, 3, \ldots, n\}, b_{2i}a \in R$, we have that for each $i \in \{2, 3, \ldots, n\}, b_{2i}a = b_{2i}as_{2i}b_{2i}a$ for some $s_{2i} \in R$. Thus from (5),

$$x = b_1 + b_{22}as_{22}b_{22}ar_2 + \ldots + b_{2n}as_{2n}b_{2n}ar_n \tag{6}$$

and

$$\begin{aligned} b_{22}as_{22}b_{22}ar_2 &= b_{22}as_{22}\left(x - b_1 - b_{23}ar_3 - \ldots - b_{2n}ar_n\right) \\ &= b_{22}as_{22}x - b_{22}as_{22}b_1 - b_{22}as_{22}b_{23}ar_3 \\ &- \ldots - b_{22}as_{22}b_{2n}ar_n. \end{aligned} \tag{7}$$

We then deduce from (6) and (7) that:

$$\begin{aligned} x &= b_1 + b_{22}as_{22}x - b_{22}as_{22}b_1 \\ &+ \left(b_{23}as_{23}b_{23} - b_{22}as_{22}b_{23}\right)ar_2 \\ &+ \ldots + \left(b_{2n}as_{2n}b_{2n} - b_{22}as_{22}b_{2n}\right)ar_n \end{aligned}.$$

But from (1) and (5):

$$\begin{aligned} &b_1 \in BaRaB, \\ &b_{22}as_{22}x \in BaRas_{22}RaB \subseteq BaRaB, \\ &b_{22}as_{22}b_1 \in BaRas_{22}BaRaB \subseteq BaRaB \end{aligned}$$

and for $i \in \{3, \ldots, n\}$,

$$\begin{aligned} &b_{2i}as_{2i}b_{2i} - b_{22}as_{22}b_{21} \\ &\in BaRas_{2i}BaR + BaRas_{22}BaR. \end{aligned}$$

Thus $b_{2i}as_{2i}b_{2i} - b_{22}as_{22}b_{21} \in BaR$, so we have:

$$x = b_2 + b_{33}ar_3 + \ldots + b_{3n}ar_n$$

for some $b_2 \in BaRaB$ and $b_{33}, \ldots, b_{3n} \in BaR$.

Continuing in this fashion, we obtain the n-1th step that:

$$x = b_{n-1} + b_{nn}ar_n \quad (8)$$

for some $b_{n-1} \in BaRaB$ and $b_{nn} \in BaR$.

Let $s_{nn} \in R$ be such that $b_{nn}a = b_{nn}as_{nn}b_{nn}a$. Then from (8):

$$x = b_{n-1} + b_{nn}as_{nn}b_{nn}ar_n \quad (9)$$

and

$$\begin{aligned} b_{nn}as_{nn}b_{nn}as_{nn}b_{nn}ar_n &= b_{nn}as_{nn}(x - b_{n-1}) \\ &= b_{nn}as_{nn}x - b_{nn}as_{nn}b_{n-1}. \end{aligned} \quad (10)$$

Thus we obtain from (9) and (10) that:

$$x = b_{n-1} + b_{nn}as_{nn}x - b_{nn}as_{nn}b_{n-1}.$$

But since by (1) and (8):

$$b_{n-1} \in BaRaB,$$
$$b_{nn}as_{nn}x \in BaRas_{nn}RaB \subseteq BaRaB \text{ and}$$
$$b_{nn}as_{nn}b_{n-1} \in BaRas_{nn}BaRaB \subseteq BaRaB,$$

it follows that $x \in BaRaB$ which implies that $x \in B$.

This proves that $RaB \cap BaR \subseteq B$, so B is a quasi-ideal of the ring (R, +, a).

Hence the theorem is proved.

DISCUSSION

It is known that every bi-ideal of regular rings is a quasi-ideal. However, although the variant of regular rings need not be a regular ring but every bi-ideal of variants of regular rings is a quasi-ideal.

CONCLUSION

Every bi-ideal of variants of regular rings is a quasi-ideal, so bi-ideals and quasi-ideals of variants of regular rings coincide.

REFERENCES

Chinram, C., 2009. Regularity-preserving elements of regular rings. Sci. Asia, 35: 111-112. DOI: 10.2306/scienceasia1513-1874.2009.35.111

Kemprasit, Y., 2002. Regularity and unit-regularity of generalized semigroups of linear transformations. Southeast Asian Bull. Math., 25: 617-622. DOI: 10.1007/s100120200006

Kemprasit Y. and S. Baupradist, 2001. A note on the multiplicative semigroups Z_n whose bi-ideals are quasi-ideals. Southeast Asian Bull. Math., 25: 269-271. DOI: 10.1007/s10012-001-0269-9

Lajos, S. and F.A. Sza'sz, 1970. On the bi-ideals in associative rings. Proc. Japan Acad., Ser. A, 46: 505-507. DOI:10.3792/pja/1195520265

Steinfeld, O., 1953. On ideal-quotients and prime ideals. Acta Math. Acad. Sci. Hung., 4: 289-298. DOI: 10.1007/BF02127587