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Hybrid Block Method for Direct Numerical Approximation of Second Order Initial Value Problems Using Taylor Series Expansions

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Abstract: In this article, a hybrid block method is utilized for the numerical approximation of second order Initial Value Problems (IVPs). The rigor of reduction to a system of first order initial value problems is bypassed as the hybrid block method directly solves the second order IVPs. Likewise, the methodology utilized also avoids the cumbersome steps involved in the widely adopted interpolation approach for developing hybrid block methods as a simple and easy to implement algorithm using the knowledge from the conventional Taylor series expansions with less cumbersome steps is introduced. To further justify the usability of this hybrid block method, the basic properties which will infer convergence when adopted to solve differential equations are investigated. The hybrid block method validates its superiority over existing methods as seen in the improved accuracy when solving the considered numerical examples.

Keywords: Hybrid Block Method, Second Order, Direct Methods, Initial Value Problems, Taylor Series

Introduction

Differential equations of the form:

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right), \; y(a) = y_0, \; \frac{dy}{dx}(a) = y_0', \; x \in [a,b] \qquad (1)$$

Are known as second order initial value problems with initial conditions prescribed at a certain point *a*. Differential equations of this form play an important role in modeling virtually every physical or biological process because such equations occur in connection with numerous problems that are encountered in various aspects of our everyday life. This concept of mathematical modeling involves translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provide insight, answers and guidance useful for the originating application (Abdelrahim and Omar, 2016; Moaddy *et al*., 2015).

Conventionally, the approach for obtaining a numerical approximation to equations of the form (1) above (also classified as higher order ordinary differential equations) usually involves a rigorous reduction of the higher order differential equations to a system of first order differential equations (James *et al*., 2013; Kayode and Adeyeye, 2011; Jator, 2007). Block multistep methods have evolved with time for the direct solution of second order initial value problems (Abdelrahim and Omar, 2016; Jator, 2007; Adesanya *et al*., 2012) and the concept of introducing evaluation at off-grid points birthing the introduction of hybrid block methods for numerical approximation of (1) have also been investigated into.

However, research continually improves upon previous by introducing new methods with better accuracy for numerically approximating differential equations. Hence, the main objective of this article which is the introduction of a new hybrid block method for numerically approximating (1) with better accuracy when compared to previously existing methods in terms of error. The methodology for developing the hybrid method is a new and easy to understand apptroach which differs from the commonly used interpolation (Jator, 2007; Adesanya *et al*., 2012; Kuboye and Omar, 2015) and numerical integration approach (See *et al*., 2016; Abdul Majid *et al*., 2012; Omar and Suleiman, 2016) for developing multistep methods.

Materials and Methods

The hybrid block method scheme for the direct numerical solution of (1) is written as:

$$y_{n+\frac{5}{6}\xi} = \sum_{i=0}^{1} \frac{\left(\frac{5}{6}\xi h\right)^i}{i!} y_n^{(i)} + h^2 \sum_{i=0}^{6} \phi_{\xi i} f_{n+\frac{5}{6}i}, \ \xi = 1,2,\cdots,6 \quad (2)$$

Recall that from the form of Equation 1 above, it is likewise needed to have an expression that computes the derivative, which is y' of the second order initial value problem under consideration. This is obtained from the block scheme defined below:

$$y'_{n+\frac{5}{6}\xi} = y'_n + h\sum_{i=0}^{6} \omega_{\xi i} f_{n+\frac{5}{6}i}, \ \xi = 1,2,\cdots,6 \quad (3)$$

Having defined the schemes in Equation 2 and 3 above, the next step is computing the values of the coefficients $\phi_{\xi i}$ and $\omega_{\xi i}$. This is derived by adopting the concept of derivation of linear multistep methods through Taylor expansions which dates back to the work of (Lambert, 1973). The approach considered the Taylor expansion for $y_{n+a} = y(x_n + ah)$ about x_n defined as:

$$y(x_n + ah) = y(x_n) + ahy^{(1)}(x_n) + \frac{(ah)^2}{2!} y^{(2)}(x_n) + \cdots \quad (4)$$

Where:

$$y^{(q)}(x_n) = \left.\frac{d^q y}{dx^q}\right|_{x=x_n}, \ q = 1,2,\cdots$$

This expression as seen in (Lambert, 1973) was limited to the derivation of multistep schemes for first order ordinary differential equations. Therefore, since this article is dealing with higher order ordinary differential equations, the Taylor expansion for $y_{n+a}^{(n)} = y^{(n)}(x_n + ah)$ about x_n is needed and defined as:

$$y^{(n)}(x_n + ah) = y^{(n)}(x_n) + ahy^{(n+1)}(x_n) + \frac{(ah)^2}{2!} y^{(n+2)}(x_n) + \cdots \quad (5)$$

Equation 4 and 5 are used for the expansion of $y_{n+\frac{5}{6}\xi}$, $f_{n+\frac{5}{6}i}$ and $y'_{n+\frac{5}{6}\xi}$ where $y_{n+ah} = f(x_n + ah) + y''(x_n + ah)$. The obtained expressions are then substituted back to Equation 2 and 3 to obtain the required scheme, however the truncation of the Taylor series expansion is with respect to the number of unknown $\phi_{\xi i}$ or $\omega_{\xi i}$.

For better clarification, consider Equation 2 with $\xi = 1$ which gives:

$$y_{n+\frac{5}{6}} = \sum_{i=0}^{1} \frac{\left(\frac{5}{6}h\right)^i}{i!} y_n^{(i)} + h^2 \sum_{i=0}^{6} \phi_{1i} f_{n+\frac{5}{6}i} \quad (6)$$

This is equivalent to the expression:

$$\begin{aligned}
&y_{n+\frac{5}{6}} = \sum_{i=0}^{1} \frac{\left(\frac{5}{6}h\right)^i}{i!} y_n^{(i)} + h^2 \sum_{i=0}^{6} \phi_{1i} f_{n+\frac{5}{6}i} \\
&\Rightarrow y(x_n) + \tfrac{5}{6}hy^{(1)}(x_n) + \frac{\left(\frac{5}{6}h\right)^2}{2!} y^{(2)}(x_n) + \frac{\left(\frac{5}{6}h\right)^3}{3!} y^{(3)}(x_n) + \frac{\left(\frac{5}{6}h\right)^4}{4!} y^{(4)}(x_n) + \frac{\left(\frac{5}{6}h\right)^5}{5!} y^{(5)}(x_n) \\
&+ \frac{\left(\frac{5}{6}h\right)^6}{6!} y^{(6)}(x_n) + \frac{\left(\frac{5}{6}h\right)^7}{7!} y^{(7)}(x_n) + \frac{\left(\frac{5}{6}h\right)^8}{8!} y^{(8)}(x_n) = y_n + \tfrac{5}{6}hy_n^{(1)} + h^2\{\phi_{10}\left(y^{(2)}(x_n)\right) \\
&+\phi_{11}(y^{(2)}(x_n) + \tfrac{5}{6}hy^{(3)}(x_n) + \frac{\left(\frac{5}{6}h\right)^2}{2!} y^{(4)}(x_n) + \frac{\left(\frac{5}{6}h\right)^3}{3!} y^{(5)}(x_n) + \frac{\left(\frac{5}{6}h\right)^4}{4!} y^{(6)}(x_n) \\
&+ \frac{\left(\frac{5}{6}h\right)^5}{5!} y^{(7)}(x_n) + \frac{\left(\frac{5}{6}h\right)^6}{6!} y^{(8)}(x_n)) + \phi_{12}(y^{(2)}(x_n) + \tfrac{10}{6}hy^{(3)}(x_n) + \frac{\left(\frac{10}{6}h\right)^2}{2!} y^{(4)}(x_n) \\
&+ \frac{\left(\frac{10}{6}h\right)^3}{3!} y^{(5)}(x_n) + \frac{\left(\frac{10}{6}h\right)^4}{4!} y^{(6)}(x_n) + \frac{\left(\frac{10}{6}h\right)^5}{5!} y^{(7)}(x_n) + \frac{\left(\frac{10}{6}h\right)^6}{6!} y^{(8)}(x_n)) + \phi_{13}(y^{(2)}(x_n) \\
&+ \tfrac{15}{6}hy^{(3)}(x_n) + \frac{\left(\frac{15}{6}h\right)^2}{2!} y^{(4)}(x_n) + \frac{\left(\frac{15}{6}h\right)^3}{3!} y^{(5)}(x_n) + \frac{\left(\frac{15}{6}h\right)^4}{4!} y^{(6)}(x_n) + \frac{\left(\frac{15}{6}h\right)^5}{5!} y^{(7)}(x_n) \\
&+ \frac{\left(\frac{15}{6}h\right)^6}{6!} y^{(8)}(x_n)) + \phi_{14}(y^{(2)}(x_n) + \tfrac{20}{6}hy^{(3)}(x_n) + \frac{\left(\frac{20}{6}h\right)^2}{2!} y^{(4)}(x_n) + \frac{\left(\frac{20}{6}h\right)^3}{3!} y^{(5)}(x_n) \\
&+ \frac{\left(\frac{20}{6}h\right)^4}{4!} y^{(6)}(x_n) + \frac{\left(\frac{20}{6}h\right)^5}{5!} y^{(7)}(x_n) + \frac{\left(\frac{20}{6}h\right)^6}{6!} y^{(8)}(x_n)) + \phi_{15}(y^{(2)}(x_n) + \tfrac{25}{6}hy^{(3)}(x_n) \\
&+ \frac{\left(\frac{25}{6}h\right)^2}{2!} y^{(4)}(x_n) + \frac{\left(\frac{25}{6}h\right)^3}{3!} y^{(5)}(x_n) + \frac{\left(\frac{25}{6}h\right)^4}{4!} y^{(6)}(x_n) + \frac{\left(\frac{25}{6}h\right)^5}{5!} y^{(7)}(x_n) + \frac{\left(\frac{25}{6}h\right)^6}{6!} y^{(8)}(x_n)) \\
&+\phi_{16}(y^{(2)}(x_n) + 5hy^{(3)}(x_n) + \frac{(5h)^2}{2!} y^{(4)}(x_n) + \frac{(5h)^3}{3!} y^{(5)}(x_n) + \frac{(5h)^4}{4!} y^{(6)}(x_n) \quad (7) \\
&+ \frac{(5h)^5}{5!} y^{(7)}(x_n) + \frac{(5h)^6}{6!} y^{(8)}(x_n))\}
\end{aligned}$$

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Equating coefficients of $h^m y^{(m)}(x_n)$ and rewriting in matrix form $Ax = B$, Equation 7 takes the form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{\left(\frac{5}{6}h\right)^1}{1!} & \frac{\left(\frac{10}{6}h\right)^1}{1!} & \frac{\left(\frac{15}{6}h\right)^1}{1!} & \frac{\left(\frac{20}{6}h\right)^1}{1!} & \frac{\left(\frac{25}{6}h\right)^1}{1!} & \frac{(5h)^1}{1!} \\ 0 & \frac{\left(\frac{5}{6}h\right)^2}{2!} & \frac{\left(\frac{10}{6}h\right)^2}{2!} & \frac{\left(\frac{15}{6}h\right)^2}{2!} & \frac{\left(\frac{20}{6}h\right)^2}{2!} & \frac{\left(\frac{25}{6}h\right)^2}{2!} & \frac{(5h)^2}{2!} \\ 0 & \frac{\left(\frac{5}{6}h\right)^3}{3!} & \frac{\left(\frac{10}{6}h\right)^3}{3!} & \frac{\left(\frac{15}{6}h\right)^3}{3!} & \frac{\left(\frac{20}{6}h\right)^3}{3!} & \frac{\left(\frac{25}{6}h\right)^3}{3!} & \frac{(5h)^3}{3!} \\ 0 & \frac{\left(\frac{5}{6}h\right)^4}{4!} & \frac{\left(\frac{10}{6}h\right)^4}{4!} & \frac{\left(\frac{15}{6}h\right)^4}{4!} & \frac{\left(\frac{20}{6}h\right)^4}{4!} & \frac{\left(\frac{25}{6}h\right)^4}{4!} & \frac{(5h)^4}{4!} \\ 0 & \frac{\left(\frac{5}{6}h\right)^5}{5!} & \frac{\left(\frac{10}{6}h\right)^5}{5!} & \frac{\left(\frac{15}{6}h\right)^5}{5!} & \frac{\left(\frac{20}{6}h\right)^5}{5!} & \frac{\left(\frac{25}{6}h\right)^5}{5!} & \frac{(5h)^5}{5!} \\ 0 & \frac{\left(\frac{5}{6}h\right)^6}{6!} & \frac{\left(\frac{10}{6}h\right)^6}{6!} & \frac{\left(\frac{15}{6}h\right)^6}{6!} & \frac{\left(\frac{20}{6}h\right)^6}{6!} & \frac{\left(\frac{25}{6}h\right)^6}{6!} & \frac{(5h)^6}{6!} \end{pmatrix} \begin{pmatrix} \phi_{10} \\ \phi_{11} \\ \phi_{12} \\ \phi_{13} \\ \phi_{14} \\ \phi_{15} \\ \phi_{16} \end{pmatrix} = \begin{pmatrix} \frac{\left(\frac{5}{6}h\right)^2}{2!} \\ \frac{\left(\frac{5}{6}h\right)^3}{3!} \\ \frac{\left(\frac{5}{6}h\right)^4}{4!} \\ \frac{\left(\frac{5}{6}h\right)^5}{5!} \\ \frac{\left(\frac{5}{6}h\right)^6}{6!} \\ \frac{\left(\frac{5}{6}h\right)^7}{7!} \\ \frac{\left(\frac{5}{6}h\right)^8}{8!} \end{pmatrix}$$

Using matrix inverse formula for system of linear equations, the value of the unknown coefficients are obtained to be:

$$\left(\phi_{10},\phi_{11},\phi_{12},\phi_{13},\phi_{14},\phi_{15},\phi_{16}\right)^T = \left(\tfrac{142745}{870912},\tfrac{6875}{20736},-\tfrac{28585}{96768},\tfrac{53105}{217728},-\tfrac{38515}{290304},\tfrac{2015}{48384},-\tfrac{4975}{870912}\right)$$

Following the same approach, the remaining unknown coefficients are obtained to be:

$$\left(\phi_{20},\phi_{21},\phi_{22},\phi_{23},\phi_{24},\phi_{25},\phi_{26}\right)^T = \left(\tfrac{5135}{13608},\tfrac{485}{378},-\tfrac{50}{81},\tfrac{985}{1701},-\tfrac{485}{1512},\tfrac{115}{1134},-\tfrac{95}{6804}\right)$$

$$\left(\phi_{30},\phi_{31},\phi_{32},\phi_{33},\phi_{34},\phi_{35},\phi_{36}\right)^T = \left(\tfrac{6325}{10752},\tfrac{4125}{1792},-\tfrac{1335}{3584},\tfrac{125}{128},-\tfrac{1815}{3584},\tfrac{285}{1792},-\tfrac{235}{10752}\right)$$

$$\left(\phi_{40},\phi_{41},\phi_{42},\phi_{43},\phi_{44},\phi_{45},\phi_{46}\right)^T = \left(\tfrac{1360}{1701},\tfrac{1880}{567},-\tfrac{10}{189},\tfrac{3280}{1701},-\tfrac{50}{81},\tfrac{40}{189},-\tfrac{50}{1701}\right)$$

$$\left(\phi_{50},\phi_{51},\phi_{52},\phi_{53},\phi_{54},\phi_{55},\phi_{56}\right)^T = \left(\tfrac{880625}{870912},\tfrac{209375}{48384},\tfrac{78125}{290304},\tfrac{640625}{217728},-\tfrac{15625}{96768},\tfrac{6875}{20736},-\tfrac{34375}{870912}\right)$$

$$\left(\phi_{60},\phi_{61},\phi_{62},\phi_{63},\phi_{64},\phi_{65},\phi_{66}\right)^T = \left(\tfrac{205}{168},\tfrac{75}{14},\tfrac{15}{28},\tfrac{85}{21},\tfrac{15}{56},\tfrac{15}{14},0\right)$$

$$\left(\omega_{10},\omega_{11},\omega_{12},\omega_{13},\omega_{14},\omega_{15},\omega_{16}\right)^T = \left(\tfrac{19087}{72576},\tfrac{2713}{3024},-\tfrac{15487}{24192},\tfrac{293}{567},-\tfrac{6737}{24192},\tfrac{263}{3024},-\tfrac{863}{72576}\right)$$

$$\left(\omega_{20},\omega_{21},\omega_{22},\omega_{23},\omega_{24},\omega_{25},\omega_{26}\right)^T = \left(\tfrac{1139}{4536},\tfrac{235}{189},\tfrac{11}{1512},\tfrac{166}{567},-\tfrac{269}{1512},\tfrac{11}{189},-\tfrac{37}{4536}\right)$$

$$\left(\omega_{30},\omega_{31},\omega_{32},\omega_{33},\omega_{34},\omega_{35},\omega_{36}\right)^T = \left(\tfrac{685}{2688},\tfrac{135}{112},\tfrac{387}{896},\tfrac{17}{21},-\tfrac{243}{896},\tfrac{9}{112},-\tfrac{29}{2688}\right)$$

$$\left(\omega_{40},\omega_{41},\omega_{42},\omega_{43},\omega_{44},\omega_{45},\omega_{46}\right)^T = \left(\tfrac{143}{567},\tfrac{232}{189},\tfrac{64}{189},\tfrac{752}{567},\tfrac{29}{189},\tfrac{8}{189},-\tfrac{4}{567}\right)$$

$$\left(\omega_{50},\omega_{51},\omega_{52},\omega_{53},\omega_{54},\omega_{55},\omega_{56}\right)^T = \left(\tfrac{18575}{72576},\tfrac{3625}{3024},\tfrac{10625}{24192},\tfrac{625}{567},\tfrac{19375}{24192},\tfrac{1175}{3024},-\tfrac{1375}{72576}\right)$$

$$\left(\omega_{60},\omega_{61},\omega_{62},\omega_{63},\omega_{64},\omega_{65},\omega_{66}\right)^T = \left(\tfrac{41}{168},\tfrac{9}{7},\tfrac{9}{56},\tfrac{34}{21},\tfrac{9}{56},\tfrac{9}{7},\tfrac{41}{168}\right)$$

Substituting all the obtained coefficients back in Equations 2 and 3 gives the hybrid block schemes:

$$\begin{aligned}
y_{n+\frac{5}{6}} &= y_n + \tfrac{5}{6}hy'_n + \tfrac{h^2}{870912}\begin{bmatrix} 142745f_n + 288750f_{n+\frac{5}{6}} - 257265f_{n+\frac{10}{6}} + 212420f_{n+\frac{15}{6}} \\ -115545f_{n+\frac{20}{6}} + 36270f_{n+\frac{25}{6}} - 4975f_{n+5} \end{bmatrix} \\
y_{n+\frac{10}{6}} &= y_n + \tfrac{10}{6}hy'_n + \tfrac{h^2}{13608}\begin{bmatrix} 5135f_n + 17460f_{n+\frac{5}{6}} - 8400f_{n+\frac{10}{6}} + 7880f_{n+\frac{15}{6}} \\ -4365f_{n+\frac{20}{6}} + 1380f_{n+\frac{25}{6}} - 190f_{n+5} \end{bmatrix} \\
y_{n+\frac{15}{6}} &= y_n + \tfrac{15}{6}hy'_n + \tfrac{h^2}{10752}\begin{bmatrix} 6325f_n + 24750f_{n+\frac{5}{6}} - 4005f_{n+\frac{10}{6}} + 10500f_{n+\frac{15}{6}} \\ -5445f_{n+\frac{20}{6}} + 1710f_{n+\frac{25}{6}} - 235f_{n+5} \end{bmatrix} \\
y_{n+\frac{20}{6}} &= y_n + \tfrac{20}{6}hy'_n + \tfrac{h^2}{1701}\begin{bmatrix} 1360f_n + 5640f_{n+\frac{5}{6}} - 90f_{n+\frac{10}{6}} + 3280f_{n+\frac{15}{6}} \\ -1050f_{n+\frac{20}{6}} + 360f_{n+\frac{25}{6}} - 50f_{n+5} \end{bmatrix} \\
y_{n+\frac{25}{6}} &= y_n + \tfrac{25}{6}hy'_n + \tfrac{h^2}{1701}\begin{bmatrix} 880625f_n + 3768750f_{n+\frac{5}{6}} + 234375f_{n+\frac{10}{6}} + 2562500f_{n+\frac{15}{6}} \\ -140625f_{n+\frac{20}{6}} + 288750f_{n+\frac{25}{6}} - 34375f_{n+5} \end{bmatrix} \\
y_{n+5} &= y_n + 5hy'_n + \tfrac{h^2}{168}\begin{bmatrix} 205f_n + 900f_{n+\frac{5}{6}} + 90f_{n+\frac{10}{6}} \\ +680f_{n+\frac{15}{6}} + 45f_{n+\frac{20}{6}} + 180f_{n+\frac{25}{6}} \end{bmatrix}
\end{aligned} \qquad (8)$$

Together with its derivatives:

$$
\begin{aligned}
y'_{n+\frac{5}{6}} &= y'_n + \tfrac{h}{870912}\begin{bmatrix} 19087f_n + 65112f_{n+\frac{5}{6}} - 46461f_{n+\frac{10}{6}} + 37504f_{n+\frac{15}{6}} \\ -20211f_{n+\frac{20}{6}} + 6312f_{n+\frac{25}{6}} - 863f_{n+5} \end{bmatrix} \\
y'_{n+\frac{10}{6}} &= y'_n + \tfrac{h}{4536}\begin{bmatrix} 1139f_n + 5640f_{n+\frac{5}{6}} + 33f_{n+\frac{10}{6}} + 1328f_{n+\frac{15}{6}} \\ -807f_{n+\frac{20}{6}} + 264f_{n+\frac{25}{6}} - 37f_{n+5} \end{bmatrix} \\
y'_{n+\frac{15}{6}} &= y'_n + \tfrac{h}{2688}\begin{bmatrix} 685f_n + 3240f_{n+\frac{5}{6}} + 1161f_{n+\frac{10}{6}} + 2176f_{n+\frac{15}{6}} \\ -729f_{n+\frac{20}{6}} + 216f_{n+\frac{25}{6}} - 29f_{n+5} \end{bmatrix} \\
y'_{n+\frac{20}{6}} &= y'_n + \tfrac{h}{567}\begin{bmatrix} 143f_n + 696f_{n+\frac{5}{6}} + 192f_{n+\frac{10}{6}} + 752f_{n+\frac{15}{6}} \\ +87f_{n+\frac{20}{6}} + 24f_{n+\frac{25}{6}} - 4f_{n+5} \end{bmatrix} \qquad (9) \\
y'_{n+\frac{25}{6}} &= y'_n + \tfrac{h}{72576}\begin{bmatrix} 18575f_n + 87000f_{n+\frac{5}{6}} + 31875f_{n+\frac{10}{6}} \\ +80000f_{n+\frac{15}{6}} + 58125f_{n+\frac{20}{6}} \\ +28200f_{n+\frac{25}{6}} - 1375f_{n+5} \end{bmatrix} \\
y'_{n+5} &= y'_n + \tfrac{h}{168}\begin{bmatrix} 41f_n + 216f_{n+\frac{5}{6}} + 27f_{n+\frac{10}{6}} + 272f_{n+\frac{15}{6}} \\ +27f_{n+\frac{20}{6}} + 216f_{n+\frac{25}{6}} + 41f_{n+5} \end{bmatrix}
\end{aligned}
$$

Results

Properties of the Method

Zero Stability

The hybrid block method is said to be zero-stable if the roots z of the first characteristic polynomial $\rho(z) = \det(zA^0 - A^1)$ satisfies $|z| \le 1$ and the root $|z| = 1$ has multiplicity not greater than the order of the differential equation (Kuboye and Omar, 2015). Therefore, the first characteristic polynomial of the hybrid block method (8) is obtained as:

$$
\det\left(zA^0 - A^1\right) = \left| z\begin{pmatrix} 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1 \end{pmatrix} - \begin{pmatrix} 0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1 \end{pmatrix} \right|
$$

Hence the roots of the polynomial are $z = 0,0,0,0,0,1$. Thus, the hybrid block method is zero-stable.

Order of the Method

Rewrite the hybrid block method in Equation 2 in its corresponding linear difference operator form as defined below:

$$
L\{y(x):h\} = y_{n+\frac{5}{6}\xi} - \sum_{i=0}^{1}\frac{\left(\frac{5}{6}\xi h\right)^i}{i!}y_n^{(i)} - h^2\sum_{i=0}^{6}\phi_{\xi i}f_{n+\frac{5}{6}i},\ \xi = 1,2,\cdots,6 \qquad (10)
$$

Following the conventional approach for obtaining the order of a linear multistep method from (Lambert, 1973), the individual terms in (8) are expanded using Taylor series expansion about x_n as defined in Equation 4. After much simplification, the hybrid block schemes in (8) are found to be of order $[7,7,7,7,7,7]^T$ with error constant of:

$$
\begin{bmatrix} 1.28841\times10^{-3} \\ 3.18568\times10^{-3} \\ 4.98360\times10^{-3} \\ 6.78153\times10^{-3} \\ 8.67879\times10^{-3} \\ 9.96720\times10^{-3} \end{bmatrix}
$$

Consistency

The hybrid block method is said to be consistent if it has an order more than or equal to one (Abdelrahim and Omar, 2016). Therefore, our method is consistent.

Convergence

The sufficient conditions for a linear multistep method to be convergent are zero stability and consistency (Henrici, 1963). Therefore since the hybrid block method is zero-stable and consistent, hence it is convergent.

Numerical Experiments

The following second order initial value problems were numerically solved for comparison of accuracy in terms of error with previously existing methods:

Problem 1:

$$
y'' = -y + 2\cos x,\ y(0) = 1,\ y'(0) = 0,\ 0 \le x \le 1
$$

Exact Solution:

$$
y(x) = \cos x + x\sin x
$$

Problem 2:

$$
y'' = y,\ y(0) = 1,\ y'(0) = 1,\ 0 \le x \le 1
$$

Exact Solution:

$$
y(x) = e^x
$$

Problems 1 and 2 above were solved by (Kuboye and Omar, 2015) using a non-hybrid six-step block method of order seven and the maximum errors within the given interval were selected. The hybrid block method was adopted to numerically solve both problems and the results obtained are compared as shown in Table 1.

Problem 3:

$$y'' = y',\ y(0) = 0,\ y'(0) = -1,\ 0 \le x \le 1$$

Exact Solution:

$$y(x) = 1 - e^x$$

Problem 3 was solved by (Mohammed *et al*., 2010) using a six-step block method of order seven. Comparison is made between their method and the new hybrid block as displayed in Table 2.

Problem 4:

$$y'' = -y,\ y(0) = 1,\ y'(0) = 1$$

Exact Solution:

$$y(x) = \cos x + \sin x$$

Awari (2013) computed the numerical solution of Problem 4 and the results with comparison to the new hybrid method is shown in Table 3.

Problem 5:

$$y'' - 4y' + 8y = x^3,\ y(0) = 2, y'(0) = 4$$

Exact Solution:

$$y(x) = \tfrac{3}{32}x + 2(\cos 2x)e^{2x} - \tfrac{3}{64}e^{2x}\sin 2x + \tfrac{3}{16}x^2 + \tfrac{1}{8}x^3$$

Problem 5 was solved using the new hybrid block in Equation 8 and comparison in terms of error was made with (Abdelrahim and Omar, 2016). Results are as displayed in Table 4.

Table 1. Comparison between new hybrid block method and (Adesanya *et al*., 2012) for Problem 1 and Problem 2

Problem 1		
h	Error (Kuboye and Omar, 2015)	Error (Hybrid block method)
0.01	1.428607E-11	8.881784E-16
0.001	1.687539E-13	2.220446E-15
Problem 2		
h	Error (Kuboye and Omar, 2015)	Error (Hybrid block method)
0.01	1.428607E-11	8.881784E-16
0.001	1.687539E-13	2.220446E-15

Table 2. Comparison between new hybrid block method and (Mohammed *et al*., 2010) for Problem 3

x	Exact solution	Computed solution	Error (Mohammed *et al*., 2010)	Error (Hybrid block method)
0.0833	-0.08690404952123	-0.08690404951952	-	1.705053E-12
0.1000	-0.10517091800000	-	5.7260E-06	-
0.1667	-0.18136041286565	-0.18136041286124	-	4.405920E-12
0.2000	-0.22140275800000	-	6.6391E-06	-
0.2500	-0.28402541668774	-0.28402541668054	-	7.198242E-12
0.3000	-0.34985880800000	-	7.0283E-06	-
0.3333	-0.39561242508609	-0.39561242507585	-	1.024042E-11
0.4000	-0.49182469800000	-	7.4539E-06	-
0.4167	-0.51689679638821	-0.51689679637454	-	1.367073E-11
0.5000	-0.64872127070013	-0.64872127068352	7.8935E-06	1.661205E-11
0.5833	-0.79200182565576	-0.79200182563489	-	2.086653E-11
0.6000	-0.82211880000000	-	8.1942E-06	-
0.6667	-0.94773404105468	-0.94773404102779	-	2.688882E-11
0.7000	-1.01375270000000	-	8.1810E-06	-
0.7500	-1.11700001661267	-1.11700001657948	-	3.319833E-11
0.8000	-1.22554092800000	-	8.1810E-06	-
0.8333	-1.30097589089283	-1.30097589085276	-	4.006750E-11
0.9000	-1.45960311100000	-	8.1730E-06	-
0.9167	-1.50094001366213	-1.50094001361439	-	4.773781E-11
1.0000	-1.71828182800000	-1.71828182840427	8.1650E-06	5.477729E-11

Table 3. Comparison between new hybrid block method and (Kayode and Adeyeye, 2011) for Problem 4

x	Exact solution	Computed solution	Error (Kayode and Adeyeye, 2011)	Error (Hybrid block method)
0.0833	1.0797667029008697	1.0797667028998994	-	9.703349E-13
0.1000	1.0948375819000000	-	1.157E-7	-
0.1667	1.1520393642563400	1.1520393642539457	-	2.394307E-12
0.2000	1.1787359086000000	-	3.099E-7	-
0.2500	1.2163163809651676	1.2163163809614408	-	3.726797E-12
0.3000	1.2508566958000000	-	5.055E-7	-
0.3333	1.2721516431108899	1.2721516431058548	-	5.035083E-12
0.4000	1.3104793363000000	-	6.957E-7	-
0.4167	1.3191576301549550	1.3191576301485788	-	6.376233E-12
0.5000	1.3570081004945758	1.3570081004873182	8.789E-7	7.257528E-12
0.5833	1.3854403557186268	1.3854403557113075	-	7.319256E-12
0.6000	1.3899780883000000	-	1.054E-6	-
0.6667	1.4042570638466851	1.4042570638392982	-	7.386980E-12
0.7000	1.4090598745000000	-	1.008E-6	-
0.7500	1.4133276288971550	1.4133276288897612	-	7.393863E-12
0.8000	1.4140628003000000	-	9.226E-7	-
0.8333	1.4125890972790938	1.4125890972717421	-	7.351675E-12
0.9000	1.4049368779000000	-	8.261E-7	-
0.9167	1.4020465947169871	1.4020465947097280	-	7.259082E-12
1.0000	1.3817732906760363	1.3817732906689242	7.216E-7	7.112089E-12

Table 4. Comparison between new hybrid block method and (Abdelrahim and Omar, 2016) for Problem 5

x	Exact solution	Computed solution	Error (Abdelrahim and Omar, 2016)	Error (Hybrid block method)
0.0833	2.3299813627353112	2.3299813454790241	-	1.725629E-08
0.1000	2.3941125770000000	-	5.7260E-06	-
0.1667	2.6375944892958487	2.6375944390018171	-	5.029403E-08
0.2000	2.7481413330000000	-	6.6391E-06	-
0.2500	2.8938356161427006	2.8938355248284888	-	9.131421E-08
0.3000	3.0078669400000000	-	7.0283E-06	-
0.3333	3.0616545199215750	3.0616543769812110	-	1.429404E-07
0.4000	3.1017624050000000	-	7.4539E-06	-
0.4167	3.0952313771655420	3.0952311710464442	-	2.061191E-07
0.5000	3.1017624050000000	2.9395428239868138	7.8935E-06	2.767584E-07

Discussion

This article has considered certain second order initial problems as highlighted in Problems 1-5. The new hybrid block method was adopted to obtain numerical approximations to these problems. As displayed in Tables 1-4, more accurate results in comparison to the exact solution is displayed by the new hybrid block method.

Conclusion

This article has introduced a new hybrid block method for the numerical approximation of second order initial value problems. To validate the superiority of the new hybrid block method, certain numerical examples were considered which have been solved previously in literature and comparison was made in terms of absolute error, that is, the difference between the exact and computed solution. The results in the tables above show improved accuracy and hence making this new hybrid method suitable for application to obtain the approximate numerical solution of real life problems modeled in the form of Equation 1 above.

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Author's Contributions

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Ethics

The authors declare no conflict of interest.

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