Strongly Real Elements in the Linear Groups PSLₙ(q)

Ibrahim A.I. Suleiman

Department of Mathematics and Statistics, Mu'tah University,
P.O.Box (7), 61710 Al-Karak, Jordan

Abstract: We investigated which elements in some projective special linear and unitary groups are strongly real. In particular we showed every real element in $PSL_2(q)$ strongly real. We write a full table for real classes which are not strongly real in the unitary groups as well as a table for the non real classes in the unitary groups.

Key words: Real elements, linear groups

INTRODUCTION

An element g in a group G is called real if g is conjugate to g^{-1} and is called strongly real if it is the product of two involutions. Similarly the conjugacy class g^G is called real or strongly real if g is. Clearly every strongly real element is real, but the converse is not true. For example, the quaternion group Q_8 has the property that all its elements are real, but its only involution is central, so cannot conjugate an element of order 4 to its inverse. In a recent study[2], Tiep and Zalesski determine which finite simple groups have the property that every element is real. A much more difficult question, raised in problem 14.82 of the Kourovka notebook[6], is which finite simple groups have the property that every element is strongly real. This problem is still open in general (but see[3,4], for some cases, and related questions). In a preprint[7], it has been determined exactly which elements in the sporadic and the alternating groups are strongly real. In the alternating groups every real element is strongly real, but this is not true in all sporadic groups. The alternating and the sporadic groups in which every element is strongly real are An for (n = 5, 6, 10, 14), and J₁and J₂. It is known[8,9]) that all real elements in the general linear group $GL_n(q)$ are strongly real. However, this is not true in the special linear groups $SL_n(q)$ for example, if n = 2, then there is only one involution in $SL_2(q)$, so only the central elements of $SL_2(q)$ are strongly real. On the other hand, we prove that all real elements in $PSL_2(q)$ are strongly real. Indeed, we conjecture that all real elements in $PSL_n(q)$ are strongly real, for arbitrary n. Moreover, we completely classify the strongly real elements in many unitary groups as far as they are in[1]. We used GAP[5] to find all elements in all these groups which are real but not strongly real.

GENERAL LINEAR GROUPS

The general linear group $GL_n(q)$ consists of all n×n matrices with entries in F_q. Equivalently it is the group of all linear automorphisms of an n-dimensional vector space over F_q. The special linear group $SL_n(q)$ is the subgroup of all matrices of determinant 1. On factoring $GL_n(q)$ and $SL_n(q)$ by the scalar matrices we get what is known as the projective general linear group $PGL_n(q)$ and projective special linear group $PSL_n(q)$. The orders of these groups are given by the following formulae:

$$|GL_n(q)| = (q-1)N, \quad |SL_n(q)|$$
$$= |PGL_n(q)| = N$$

and

$$|PL_n(q)| = |L_n(q)| = \frac{N}{d}$$

where,

$$N = q^{n(n-1)/2}(q^n - 1)....(q^2 - 1)$$

and d is the greatest common divisor of q-1 and n.

Projective special linear groups ln (q): We know from[2], that the only groups $L_n(q)$ in which all elements are real are the groups $L_n(q)$ with either $q = 2^k$ or $q \equiv 1 \mod 4$. Here we show that in fact all

Current Address: Ibrahim A.I. Suleiman, Department of Mathematical and Physical Sciences, University of Nizwa, Birkat ALMouz, P.O. Box 33, 616 Nizwa, Sultanate of Oman

elements in these groups are strongly real. We also determine which elements are strongly real in $L_n(q)$ for $q \equiv 3 \bmod 4$.

Proposition 1: In this case the elements have order dividing $L_2(2^k)$, where $k > 1$, are strongly real.

Proof: The elements of $L_2(2^k)$ have order dividing 2, or $2^k - 1$, or $2^k + 1$. Indeed the normalizers of the cyclic groups of orders 2, $2^k - 1$, and $2^k + 1$ are C_2^k, $D_{2(2^k - 1)}$ and $D_{2(2^k + 1)}$ respectively. Since $C_2^k > (C_2 \times C_2) = D_4$, the result follows.

Proposition 2: All elements in $L_n(q)$, where $q \equiv 1 \bmod 4$, are strongly real.

Proof: In this case the elements of order divide either p (where p is the prime dividing q), or $(q-1)/2$ or $(q+1)/2$. In the last two cases, the normalizers of the corresponding cyclic groups are now D_{q-1} and D_{q+1}, so all these classes are strongly real. In the first case, the normalizer of the cyclic group is contained in the Borel subgroup $C_p^k : C_{(q-1)}$, where $q = p^k$. Because $q \equiv 1 \bmod 4$, this subgroup contains $C_p^k : C_2$, in which the C_2 conjugates all elements in the C_p^k to their inverses. Therefore all these elements are strongly real also.

Proposition 3: All real elements in $L_n(q)$, where $q \equiv 3 \bmod 4$, are strongly real.

Proof: In this case the elements of order p, (where p is the prime dividing q) are not real. All other elements have order dividing $(q-1)/2$ or $(q+1)/2$. The normalizers of the corresponding cyclic groups are again D_{q-1} and D_{q+1}, so all these classes are strongly real.

Unitary Groups: The projective general unitary group $PGU_n(q)$ is the subgroup of all elements of GL (q^2) that fix the corresponding non-singular conjugate-symmetric sesquilinear form. The determinant of a unitary matrix is necessarily a $(q+1)^{st}$ of unity. The projective general unitary group $PSU_n(q)$ is the subgroup of unitary matrices of determinant 1. On factoring $PGU_n(q)$ and $PSU_n(q)$ by the scalar matrices we get what is known as the special unitary group $PGU_n(q)$ and the projective special unitary group $PSU_n(q)$. The orders of these groups are given by the following formulae:

$$\left| GU_n(q) \right| = (q+1)N, \quad \left| SU_n(q) \right| = \left| PGU_n(q) \right| = N$$

and

$$\left| PSU_n(q) \right| = \left| U_n(q) \right| = \frac{N}{d}$$

where,

$$N = q^{n(n-1)/2}(q^n - (-1)^n).....(q^3 + 1)(q^2 - 1)$$

and d is the greatest common divisor of q+1 and n.

In our case for the unitary groups, we can check using the character table whether each element is a product of two involutions. For element z in a conjugacy class Z is a product of elements x and y in classes X and Y respectively if and only if the structure constant:

$$\xi(X, Y, Z) = \frac{|G|}{|C_G(x)||C_G(y)|} \sum_\chi \frac{\chi(x)\chi(y)\overline{\chi(z)}}{\chi(1)}$$

is non zero. Indeed $\xi(X, Y, Z)$ is well-known to be equal to the number of ways that z can be expressed as a product of xy with x in X and y in Y. Thus to determine if z is a product of two involutions, we only need to add up the structure constants $\xi(X, Y, Z)$ over classes X, Y of involutions, and we see whether the resulting integer is positive or zero. In the study[2], Tiep and Zalesski determine that unitary groups have non-real elements. In this section we completely classify the strongly real elements in all unitary groups as far as they are in[1]. Table 1 gives us all elements which are real but not strongly real. For completeness, we also list all the non real elements in the unitary groups. Table 2 is the complete list for all these non real elements.

Table 1: Real classes which are not strongly real in unitary groups

U3 (3)	3A, 6A.
U4 (2)	all real elements are strongly real.
U3 (4)	4A
U3 (5)	5A, 10A.
U4 (3)	3A, 6A, 8A, 12A.
U3 (9)	3A, 6A.
U3 (11)	11A, 22A.
U6 (2)	8A
U5 (2)	all real elements are strongly real.
U3 (7)	all real elements are strongly real.
U3 (8)	all real elements are strongly real.

Table 2: Non- Real classes in unitary groups

U3 (3)	4AB, 7AB, 8AB, 12AB
U4 (2)	3AB, 6AB, 6CD, 9AB, 12AB
U3 (4)	5ABCD, 10ABCD, 13ABCD, 15ABCD
U3 (5)	7AB, 8AB
U4 (3)	7AB, 9ABCD
U3 (9)	5ABCD, 10ABCD, 10G- N, 20F . . . 80-P
U3 (11)	4AB, 8AB, 10BCD, 12AB, 20ABCD, 37A -M, 40A -H, 4AB
U6 (2)	6AB,9AB, 11AB, 12ABDE, 18AB

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