American Journal of Applied Sciences 4 (6): 371-373, 2007
ISSN 1546-9239

A Fixed Point Theorem for Contraction Type Mappings in Menger Spaces

Servet Kutukcu
Department of Mathematics, Faculty of Science and Arts, Ondokuz Mayis University
Kurupelit, 55139 Samsun, Turkey

Abstract: We proved a common fixed point theorem for a sequence of self maps satisfying a new contraction type condition in Menger spaces, results extended and generalize some known results in metric spaces and fuzzy metric spaces.

Key words: Fixed point, contraction map, Menger probabilistic metric space

INTRODUCTION

There have been a number of generalizations of metric space. One such generalization is Menger space introduced in 1942 by Menger[1] who was use distribution functions instead of nonnegative real numbers as values of the metric. Schweizer and Sklar[2] studied this concept and gave some fundamental results on this space. The important development of fixed-point theory in Menger spaces was due to Sehgal and Bharucha-Reid[3]. The study of common fixed points of maps satisfying some contractive type condition has been at the centre of vigorous research activity. It is observed by many authors[3,4-10] that contraction condition in metric space may be translated into probabilistic metric space endoved with min norms. The purpose of this was to define and investigate a new class of self-maps satisfying a new contraction type condition in Menger spaces.

Preliminaries: We recall some definitions and known results in Menger probabilistic metric space. For more details, we refer the readers to[1,4-9,11,12].

Definition 1: A triangular norm $*$ (shorty t-norm) is a binary operation on the unit interval [0,1] such that for all $a, b, c, d \in [0,1]$ the following conditions are satisfied:
(a) $a * 1 = a$,
(b) $a * b = b * a$,
(c) $a * b \le c * d$ whenever $a \le c$ and $b \le d$,
(d) $a * (b * c) = (a * b) * c$.
Some examples of t-norms are $a * b = \max\{a+b-1,0\}$ and $a * b = \min\{a,b\}$.

Definition 2: A distribution function is a function $F:[-\infty,\infty]\to[0,1]$ which is left continuous on $\Re$, non-decreasing and $F(-\infty) = 0$, $F(\infty) = 1$. If X is a nonempty set, $F: X\times X \to \Delta$ is called a probabilistic distance on X and $F(x,y)$ is usually detoned by F_{xy}.

Definition 3 ([11]): (*see also[1-3,9]*) The ordered pair (X,F) is called a probabilistic semimetric space (shortly PSM-space) if X is a nonempty set and F is a probabilistic distance satisfying the following conditions: for all $x, y, z \in X$ and $t, s > 0$,
(PM-1) $F_{xy}(t) = H(t) \Leftrightarrow x = y,$
(PM-2) $F_{xy} = F_{yx}$.
If, in addition, the following inequality takes place:
(PM-3) $F_{xz}(t) = 1$, $F_{zy}(s) = 1 \Rightarrow F_{xy}(t+s) = 1$, then (X,F) is called a probabilistic metric space.

The ordered triple $(X,F,*)$ is called Menger probabilistic metric space (shortly Menger space) if (X,F) is a PM-space, $*$ is a t-norm and the following condition is also satisfies: for all $x, y, z \in X$ and $t, s > 0$, (PM-4) $F_{xy}(t+s) \ge F_{xz}(t) * F_{zy}(s)$. For every PSM-space (X,F), we can consider the sets of the form $U_{\varepsilon,\lambda} = \{(x,y) \in X\times X : F_{xy}(\varepsilon) > 1-\lambda\}$.

The family $\{U_{\varepsilon,\lambda}\}_{\varepsilon>0,\lambda\in(0,1)}$ generates a semi uniformity denoted by U_F and a topology τ_F called the F-topology or the strong topology. Namely, $A \in \tau_F$ iff $\forall x \in A\ \exists\varepsilon > 0$ and $\lambda \in (0,1)$ such that $U_{\varepsilon,\lambda}(x) \subset A$. U_F is also generated by the family $\{V_\delta\}_{\delta>0}$ where $V_\delta := U_{\delta,\delta}$ ([2]).

In [13], it is proved if $\sup_{t<0} (t * t) = 1$, then U_F is a uniformity, called F-uniformity, which is metrizable. The F-topology is generated by the F-uniformity and is determined by the F-convergence: $x_n \to x$

Corresponding Author: Servet Kutukcu, Department of Mathematics, Faculty of Science and Arts, Ondokuz Mayis University, Kurupelit, 55139 Samsun, Turkey

$\Leftrightarrow F_{x_n x}(t) \to 1, \forall t > 0$.

Definition 4 ([2]): A sequence $\{x_n\}$ in a Menger space $(X,F,*)$ is called converge to a point x in X (written as $x_n \to x$) if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is an integer $n_0 = n_0(\varepsilon,\lambda)$ such that $F_{x_n x}(\varepsilon) > 1-\lambda$ for all $n \geq n_0$. The sequence called Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there is an integer $n_0 = n_0(\varepsilon,\lambda)$ such that $F_{x_n x_m}(\varepsilon) > 1-\lambda$ for all $n, m \geq n_0$. A Menger space $(X,F,*)$ is said to be complete if every Cauchy sequence in it converges to a point of it.

Lemma 1 ([9]): Let $\{x_n\}$ be a sequence in a Menger space $(X,F,*)$ with continuous t- norm $*$ and $t*t \geq t$. If there exists a constant $\alpha \in (0,1)$ such that $F_{x_n x_{n+1}}(\alpha t) \geq F_{x_{n-1} x_n}(t)$ for all $t > 0$ and $n = 1,2...$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2 ([9]): Let $(X,F,*)$ be a Menger space. If there exists a constant $\alpha \in (0,1)$ such that $F_{xy}(\alpha t) \geq F_{xy}(t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

Remark 1: In a Menger space $(X,F,*)$, if $t*t \geq t$ for all $t \in [0,1]$ then $a*b = \min\{a,b\}$ for all $a, b \in [0,1]$ and it is well known that such t-norm is continuous.

RESULTS

Theorem 1: Let $\{T_n\}$, n = 1, 2, ... be a sequence of mappings of a complete Menger space $(X,F,*)$ into itself with $t*t \geq t$ for all $t \in [0,1]$ and $S : X \to X$ be a continuous mapping such that $T_n(X) \subseteq S(X)$ and S is commuting with each T_n. If there exists a constant $\alpha \in (0,1)$ such that for any two mappings T_i and T_j

$$\min\{ F^2_{T_i x T_j y}(\alpha t), F_{SxT_i x}(\alpha t) F_{SyT_j y}(\alpha t), F^2_{SyT_j y}(\alpha t) \} + a\, F_{SyT_j y}(\alpha t)\ F_{SxT_j y}(2\alpha t) \geq [p\ F_{SxT_i x}(t) + q\ F_{SxSy}(t)]\ F_{SxT_j y}(2\alpha t)$$

holds for all $x, y \in X$ and $0 < p,q < 1$ and $0 \leq a < 1$ such that $p+q-a = 1$, then there exists a unique common fixed point for all T_n and S.

Proof: Let x_0 be an arbitrary point of X and $\{x_n\}$ be a sequence defined by $Sx_n = T_n x_{n-1}$, $n = 1,2,...$ Then for each $t > 0$ and $0 < \alpha < 1$, we have

$$\min\{ F^2_{T_1 x_0 T_2 x_1}(\alpha t), F_{Sx_0 T_1 x_0}(\alpha t) F_{Sx_1 T_2 x_1}(\alpha t), F^2_{Sx_1 T_2 x_1}(\alpha t) \} + a\, F_{Sx_1 T_2 x_1}(\alpha t)\ F_{Sx_0 T_2 x_1}(2\alpha t) \geq$$

$$[p\, F_{Sx_0 T_1 x_0}(t) + q\, F_{Sx_0 Sx_1}(t)]\, F_{Sx_0 T_2 x_1}(2\alpha t) \text{ and}$$

$$\min\{ F^2_{Sx_1 Sx_2}(\alpha t), F_{Sx_0 Sx_1}(\alpha t) F_{Sx_1 Sx_2}(\alpha t), F^2_{Sx_1 Sx_2}(\alpha t) \} + a\, F_{Sx_1 Sx_2}(\alpha t)\ F_{Sx_0 Sx_2}(2\alpha t) \geq [p\, F_{Sx_0 Sx_1}(t) + q\, F_{Sx_0 Sx_1}(t)]\, F_{Sx_0 Sx_2}(2\alpha t).$$

Thus, it follows that

$$\min\{ F^2_{Sx_1 Sx_2}(\alpha t), F_{Sx_0 Sx_1}(\alpha t) F_{Sx_1 Sx_2}(\alpha t) \} + a\, F_{Sx_1 Sx_2}(\alpha t)\ F_{Sx_0 Sx_2}(2\alpha t) \geq (p+q)\, F_{Sx_0 Sx_1}(t)\ F_{Sx_0 Sx_2}(2\alpha t) \text{ and}$$

$$F_{Sx_1 Sx_2}(\alpha t) \min\{ F_{Sx_1 Sx_2}(\alpha t), F_{Sx_0 Sx_1}(\alpha t) \} + a\, F_{Sx_1 Sx_2}(\alpha t)\ F_{Sx_0 Sx_2}(2\alpha t) \geq (p+q)\, F_{Sx_0 Sx_1}(t)\ F_{Sx_0 Sx_2}(2\alpha t).$$

Since $F_{Sx_0 Sx_2}(2\alpha t) \geq \min\{ F_{Sx_0 Sx_1}(\alpha t), F_{Sx_1 Sx_2}(\alpha t) \}$, we have

$$F_{Sx_1 Sx_2}(\alpha t)\ F_{Sx_0 Sx_2}(2\alpha t) + a\, F_{Sx_1 Sx_2}(\alpha t)\ F_{Sx_0 Sx_2}(2\alpha t) \geq (p+q)\, F_{Sx_0 Sx_1}(t)\ F_{Sx_0 Sx_2}(2\alpha t) \text{ and}$$

$$(1+a)\, F_{Sx_1 Sx_2}(\alpha t)\ F_{Sx_0 Sx_2}(2\alpha t) \geq (p+q)\, F_{Sx_0 Sx_1}(t)\ F_{Sx_0 Sx_2}(2\alpha t).$$

Since $p + q - a = 1$, we have $F_{Sx_1 Sx_2}(\alpha t) \geq F_{Sx_0 Sx_1}(t)$. By induction, $F_{Sx_n Sx_{n+1}}(\alpha t) \geq F_{Sx_{n-1} Sx_n}(t), n = 1,2,...$ Thus, by Lemma 1, $\{Sx_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $Sx_n \to u$. Since $Sx_n = T_n x_{n-1}$, $\{T_n x_{n-1}\}$ also converges to u. Since S commutes with each T_n, using (3.1), we have

$$\min\{ F^2_{SSx_n T_k u}(\alpha t), F_{SSx_{n-1} SSx_n}(\alpha t) F_{SuT_k u}(\alpha t), F^2_{SuT_k u}(\alpha t) \} + a\, F_{SuT_k u}(\alpha t)\ F_{SSx_{n-1} T_k u}(2\alpha t) \geq [p\, F_{SSx_{n-1} SSx_n}(t) + q\, F_{SSx_{n-1} Su}(t)]\, F_{SSx_{n-1} T_k u}(2\alpha t).$$

Using the continuity of S and taking limits on both sides, we have

$$\min\{ F^2_{SuT_k u}(\alpha t), F_{SuSu}(\alpha t) F_{SuT_k u}(\alpha t), F^2_{SuT_k u}(\alpha t) \} + a\, F_{SuT_k u}(\alpha t)\ F_{SuT_k u}(2\alpha t) \geq$$

$[p\, F_{SuSu}(t) + q\, F_{SuSu}(t)]\, F_{SuT_k u}(2\alpha t)$ and so $F^2_{SuT_k u}(\alpha t) + a\, F_{SuT_k u}(\alpha t)\ F_{SuT_k u}(2\alpha t) \geq (p+q)\, F_{SuT_k u}(2\alpha t)$.

Since $F_{SuT_k u}(2\alpha t) \geq \min\{ F_{SuSu}(\alpha t), F_{SuT_k u}(\alpha t) \} = F_{SuT_k u}(\alpha t)$, we have

$$(1+a)\, F^2_{SuT_k u}(2\alpha t) = F^2_{SuT_k u}(2\alpha t) + a\, F_{SuT_k u}(2\alpha t)\ F_{SuT_k u}(2\alpha t) \geq (p+q)\, F_{SuT_k u}(2\alpha t)$$

and hence $F_{SuT_k u}(2\alpha t) \geq 1$ for all $\alpha \in (0,1)$ and $t > 0$. Therefore $Su = T_k u$ for any fixed integer k. Moreover,

$$\min\{ F^2_{Sx_n T_k u}(\alpha t), F_{Sx_{n-1} Sx_n}(\alpha t) F_{SuT_k u}(\alpha t), F^2_{SuT_k u}(\alpha t) \} +$$

$$a\, F_{SuT_k u}(\alpha t)\; F_{Sx_{n-1}T_k u}(2\alpha t) \geq$$
$$[p\, F_{Sx_{n-1}Sx_n}(t) + q\, F_{Sx_{n-1}Su}(t)]\, F_{Sx_{n-1}T_k u}(2\alpha t).$$

Taking the limits on both sides, we have

$$\min\{ F^2_{uT_k u}(\alpha t), F_{uu}(\alpha t)F_{SuSu}(\alpha t), F^2_{SuSu}(\alpha t) \} + a\, F_{SuSu}(\alpha t)$$
$$F_{uT_k u}(2\alpha t) \geq [p\, F_{uu}(t) + q\, F_{uSu}(t)]\, F_{uT_k u}(2\alpha t)$$

and so

$$F^2_{uT_k u}(\alpha t) + a\; F_{uT_k u}(2\alpha t) \geq [p + q\; F_{uSu}(t)]\;\; F_{uT_k u}(2\alpha t).$$

Thus, it follows that $F_{uT_k u}(2\alpha t) \geq 1$ for all $\alpha \in (0,1)$ and $t > 0$. Therefore $u = Su = T_k u$ for any fixed integer k. Thus u is a common fixed point of S and T_n for $n = 1, 2, \ldots$

For uniquenesses, let v be another common fixed point of S and T_n for $n = 1, 2, \ldots$ Using (3.1), we have

$$\min\{ F^2_{uv}(\alpha t), F_{Suu}(\alpha t)F_{Svv}(\alpha t), F^2_{Svv}(\alpha t) \} +$$
$$a\, F_{Svv}(\alpha t)\; F_{Suv}(2\alpha t) \geq$$
$$[p\, F_{Suu}(t) + q\, F_{SuSv}(t)]\, F_{Suv}(2\alpha t) \text{ and}$$
$$F^2_{uv}(\alpha t) + a\, F_{uv}(2\alpha t) \geq [p + q\, F_{uv}(t)]\, F_{uv}(2\alpha t).$$

So $F_{uv}(2\alpha t) \geq 1$ for all $\alpha \in (0,1)$ and $t > 0$. Hence, by Lemma 2, $u = v$. This completes the proof. If we take $a = 0$ in the main Theorem, we have the following:

Corollary 1: Let $\{T_n\}$, n = 1,2, ... be a sequence of mappings of a complete Menger space $(X, F, *)$ into itself with $t * t \geq t$ for all $t \in [0,1]$ and $S : X \to X$ be a continuous mapping such that $T_n(X) \subseteq S(X)$ and S is commuting with each T_n. If there exists a constant $\alpha \in (0,1)$ such that for any two mappings T_i and T_j

$$\min\{ F^2_{T_i x T_j y}(\alpha t), F_{SxT_i x}(\alpha t)F_{SyT_j y}(\alpha t), F^2_{SyT_j y}(\alpha t) \} \geq$$
$$[p\, F_{SxT_i x}(t) + q\, F_{SxSy}(t)]\, F_{SxT_j y}(2\alpha t)$$

holds for all $x, y \in X$ and $0 < p,q < 1$ such that $p+q = 1$, then there exists a unique common fixed point for all T_n and S.

Proof: It is easy to verify from Theorem 1. If we take $a = 0$ and $S = I_X$ (the identity map on X) in the main Theorem, we have the following:

Corollary 2: Let $\{T_n\}$, n = 1,2, ... be a sequence of mappings of a complete Menger space $(X, F, *)$ into itself with $t * t \geq t$ for all $t \in [0,1]$. If there exists a constant $\alpha \in (0,1)$ such that for any two mappings T_i and T_j

$$\min\{ F^2_{T_i x T_j y}(\alpha t), F_{xT_i x}(\alpha t)F_{yT_j y}(\alpha t), F^2_{yT_j y}(\alpha t) \} \geq$$
$$[p\, F_{xT_i x}(t) + q\, F_{xy}(t)]\, F_{xT_j y}(2\alpha t)$$

holds for all $x, y \in X$ and $0 < p,q < 1$ such that $p+q = 1$, then for any $x_0 \in X$ the sequence $\{x_n\} = \{T_n x_{n-1}\}$, $n = 1, 2, \ldots$ converges and its limit is the unique common fixed for all T_n.

Proof: Existance and uniquess of common fixed point follows from Theorem 1. Convergence of the sequence $\{x_n\}$ can be proved as in Theorem 1.

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