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On Concircular Structure Spacetimes II

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Abstract: We studied concircular structure spacetimes which are connected 4-dimensional Lorentzian concircular structure manifolds.

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INTRODUCTION

Recently the first author introduced the notion of Lorentzian concircular structure manifolds (briefly $(LCS)_n$-manifolds) by citing an example of dimension 4[6]. Then in[7] the present authors studied its several applications to general relativity and physics. In this study we extend the study of[7] and investigate some other interesting applications to relativity and cosmology. After preliminaries, we study perfect fluid non-flat $(CS)_4$-spacetimes and proved that if in such a spacetime the square of the length of the Ricci-operator is $(1/3)r^2$, then the spacetime can not contain pure matter and also in such a spacetime the pressure of the fluid is positive for $\alpha^2>\rho$ and negative for $\alpha^2>\rho$, α, ρ being non-zero scalars associated with the $(CS)_4$-spacetime. Section 4 is concerned with $(CS)_4$-spacetimes whose energy-momentum tensor is a Codazzi tensor and it is shown that in such a spacetime both the energy density and pressure of the fluid are constants over a hypersurface. Among others it is proved that if the energy-momentum tensor of a perfect fluid $(CS)_4$-spacetime is a Codazzi tensor, then the possible local cosmological structure of the spacetime is of Petrov type I, D or O and also it is shown that if a perfect fluid $(CS)_4$-spacetime with divergence-free conformal curvature tensor admits a conformal Killing vector field then the spacetime is either conformally flat or of Petrov type N. The last section deals with a conformally flat $(CS)_4$-spacetime and proved that such a spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.

Preliminaries: An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, \ldots, +)$, where $T_p M$ denotes the tangent vector space of M at p and R is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, $= 0$, > 0)[3]. The category to which a given vector falls is called its causal character.

Let M^n be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have:

$$g(\xi,\xi) = -1 \tag{1}$$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that for:

$$g(X, \xi) = \eta(X) \tag{2}$$

the equation of the following form holds:

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\,\eta(Y)\} \quad (\alpha \neq 0) \tag{3}$$

for all vector fields X,Y where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies:

$$\nabla_x \alpha = (X\alpha) = \alpha(X) = \rho\,\eta(X) \tag{4}$$

ρ being a certain scalar function. If we put:

$$\phi X = \frac{1}{\alpha}\nabla_X\xi \tag{5}$$

then from (3) and (5) we have:

$$\phi X = X + \eta(X)\,\xi, \tag{6}$$

from which it follows that ϕ is a symmetric (1,1) tensor. Thus the Lorentzian manifold M^n together with the unit timelike concircular vector field ξ, its associated 1-form η and (1,1) tensor field ϕ is said to

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be a Lorentzian concircular structure manifold (briefly $(LCS)_n$-manifold)[4]. In a $(LCS)_n$-manifold, the following relations hold[4]:

$$\text{a) } \eta(\xi) = -1, \text{ b) } \phi\xi = 0, \text{ c) } \eta(\phi X) = 0,$$
$$\text{d) } g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad (7)$$

$$\eta(R(X,Y)Z) = (\rho-\alpha^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] \quad (8)$$

$$S(X,\xi) = (n-1)(\rho-\alpha^2)\eta(X) \quad (9)$$

$$R(X,Y)\xi = (\rho-\alpha^2)[\eta(Y)X - \eta(X)Y] \quad (10)$$

for any vector fields X, Y, Z where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold. General relativity flows from Einstein's equation given by:

$$S(X,Y) - (r/2)g(X,Y) + \lambda g(X,Y) = kT(X,Y) \quad (11)$$

for all vector fields X, Y where S is the Ricci tensor of the type (0,2), r is the scalar curvature, λ is the cosmological contant, k is the gravitational constant and T is the energy momentum tensor of type (0,2).
The energy momentum tensor T is said to describe a perfect fluid[3] if

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) \quad (12)$$

where σ is the energy density function, p is the isotropic pressure function of the fluid, A is a non-zero 1-form such that g(X,U) = A(X) for all X, U being the flow vector field of the fluid.
In a $(CS)_4$-spacetime by considering the characteristic vector field ξ ξ of the spacetime as the flow vector field of the fluid, the energy momentum tensor takes the form:

$$T(X,Y) = (\sigma + p)\eta(X)\eta(Y) + pg(X,Y) \quad (13)$$

The above results will be used in the next sections.

Perfect fluid non-flat $(CS)_4$-spacetimes: In this section we consider that the matter distribution of a non-flat $(CS)_4$-spacetime be perfect fluid with σ and p as its density and pressure respectively and the characteristic vector field ξ of the spacetime as the flow vector field of the fluid. We take Einstein's field equation without cosmological constant. Then (11) can be written as:

$$S(X,Y) - (r/2)g(X,Y) = kT(X,Y) \quad (14)$$

From (13) and (14) we have:

$$S(X,Y) - (r/2)g(X,Y) = k[(\sigma+p)\eta(X)\eta(Y) + pg(X,Y)] \quad (15)$$

Taking a frame field and contracting (15) over X and Y we obtain:

$$r = k(\sigma - 3p) \quad (16)$$

In view of (16), (15) yields:

$$S(X,Y) = k\begin{bmatrix}(\sigma+p)\eta(X)\eta(Y) \\ +\frac{1}{2}(\sigma-p)g(X,Y)\end{bmatrix} \quad (17)$$

Let Q be the Ricci operator i.e., g(QX,Y)=S(X,Y). Then setting X = QX in (17) we get:

$$S(QX,Y) = k\begin{bmatrix}(\sigma+p)\eta(QX)\eta(Y) \\ +\frac{1}{2}(\sigma-p)S(X,Y)\end{bmatrix} \quad (18)$$

Contracting (18) over X and Y we have:

$$\|Q\|^2 = k\left[(\sigma+p)S(\xi,\xi) + \frac{1}{2}(\sigma-p)r\right] \quad (19)$$

Using (16) and (9) (for n = 4) in (19) we obtain:

$$\|Q\|^2 = k\begin{bmatrix}(\sigma+p)(-3)(p-\alpha^2) \\ +\frac{1}{2}(\sigma-p)k(\sigma-3p)\end{bmatrix} \quad (20)$$

Again setting X = Y = ξ in (17) we get:

$$-3(p-\alpha^2) = \frac{k}{2}(\sigma+3p) \quad (21)$$

Since the $(CS)_4$-spacetime under consideration is non-flat, we have $(\rho-\alpha^2)\neq 0$ and hence, (21) implies that: $(\sigma+3p)\neq 0$ as $k\neq 0$ By virtue of (21) we obtain from (20) that:

$$\|Q\|^2 = k^2(\sigma^2 + 3p^2) \quad (22)$$

We now suppose that the length of the Ricci operator of the perfect fluid non-flat $(CS)_4$-spacetime is $(1/3)r^2$, where r is the scalar curvature of the spacetime. Then from (22) we have:

$$\frac{1}{3}r^2 = k^2(\sigma^2 + 3p^2)$$

which yields by virtue of (16) that $k^2\sigma(\sigma+3p) = 0$ Since $\sigma+3p \neq 0$ and $k\neq 0$, it follows that $\sigma=0$ which is not possible as when the pure matter exists σ is always greater than zero. Hence the spacetime under consideration cannot contain pure matter.

Now we determine the sign of pressure in such a spacetime without pure matter. Hence for $\sigma=0$, (16) implies that:

$$p = -\frac{r}{3k} \tag{23}$$

Again for $\sigma= 0$, (15) yields $r=6(\rho-\alpha^2)$ Therefore (23) reduces to

$$p = -\frac{2}{k}(\rho - \alpha^2)$$

This implies that $p>0$ if $\alpha^2>\rho$ and $p<0$ if $\alpha^2<\rho$. Thus we can state the following:

Theorem 1: If a perfect fluid non-flat $(CS)_4$-spacetime obeying Einstein's equation without cosmological constant and the square of the length of the Ricci operator is $(1/3)r^2$, then the spacetime can not contain pure matter. Moreover in such a spacetime without pure matter the pressure of the fluid is positive or negative according as:

$$\alpha^2 > \rho \text{ or } \alpha^2<\rho$$

$(CS)_4$-spacetimes whose energy-momentum tensor is a codazzi tensor: This section deals with a $(CS)_4$-spacetime whose energy-momentum tensor T is a Codazzi tensor. Then we have:

$$(\nabla_X T)(Y,Z) = (\nabla_Z T)(Y,X) \tag{24}$$

We take Einstein's equation with cosmological constant given by (11). Then differentiating (11) covariantly we get:

$$(\nabla_X S)(Y,Z) - \frac{1}{2}dr(X)g(Y,Z) = k(\nabla_X T)(Y,Z) \tag{25}$$

This implies

$$\begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y) - \frac{1}{2}dr(X)g(Y,Z)\\ + \frac{1}{2}dr(Z)g(Y,X) = k\left[(\nabla_X T)(Y,Z) - (\nabla_Z T)(Y,X)\right]\end{aligned} \tag{26}$$

By virtue of (24) and (26) we get:

$$\begin{aligned}(\nabla_X S)(Y,Z) - (\nabla_Z S)(X,Y)\\ - \frac{1}{2}dr(X)g(Y,Z) + \frac{1}{2}dr(Z)g(X,Y) = 0\end{aligned} \tag{27}$$

Taking a frame field and contracting (27) over Y and Z, we obtain:

$$dr(X) = 0 \quad \text{for all } X \tag{28}$$

Using (28) in (27) we have:

$$(\nabla_X S)(Y,Z) = (\nabla_Z S)(X,Y) \tag{29}$$

This leads to the following:

Theorem 2: If a $(CS)_4$-spacetime has a Codazzi type of energy-momentum tensor, then its scalar curvature is constant and its Ricci tensor is of Codazzi type.

Let $T(X,Y)=g(\tilde{T}X,Y)$. Then from (11), it follows that:

$$QX = \frac{1}{2}rX + k\tilde{T}X - \lambda X \tag{30}$$

where Q is the Ricci operator. Then (24) can be written as:

$$(\nabla_X \tilde{T})(Y) = (\nabla_Y \tilde{T})(X) \tag{31}$$

From (13) we have:

$$\tilde{T}X = (\sigma + p)\eta(X)\xi + pX \tag{32}$$

Differentiating (32) covariantly we get:

$$\begin{aligned}(\nabla_X \tilde{T})(Y) = (X\sigma + Xp)\eta(Y)\xi\\ + (\sigma+p)(\nabla_X\eta)(Y)\xi\\ + (Xp)Y + (\sigma+p)\eta(Y)\nabla_X\xi\end{aligned} \tag{33}$$

In view of (33) we obtain by virtue of (31) that:

$$\begin{aligned}(X\sigma + Xp)\eta(Y)\xi + \alpha(\sigma+p)\eta(Y)X\\ +(Xp)Y - (Y\sigma + Yp)\eta(X)\xi\\ -\alpha(\sigma+p)\eta(X)\phi Y - (Yp)X = 0\end{aligned} \tag{34}$$

where (3) have been used.
Setting $Y=\xi$ in (34) and then using (7) we get:

$$\alpha(\sigma+p)\phi X = -(X\sigma)\xi - (\xi\sigma + \xi p)\eta(X)\xi - (\xi p)X \tag{35}$$

Contracting (30) we obtain:

$$r = 4\lambda + (\sigma - 3p)k \tag{36}$$

Differentiating (36) covariantly along X we have:

$$dr(X) = \left(X\sigma - 3(Xp)\right)k \quad (37)$$

Since the spacetime under consideration has Codazzi type energy-momentum tensor, we have the relation (28). By virtue of (28) and (37) we get:

$$(Xp) = \frac{1}{3}(X\sigma) \quad (38)$$

Using (38) in (35) we obtain:

$$\alpha(\sigma + p)\phi X = -3(Xp)\xi - 4(\xi p)\eta(X)\xi - (\xi p)X \quad (39)$$

Taking the inner product on both sides of (39) by ξ we get by virtue of (7) that:

$$Xp = -(\xi p)\eta(X) \quad (40)$$

From (38) and (40), it follows that:

$$X\sigma = -(\xi\sigma)\eta(X) \quad (41)$$

Again from (40) and (41) we have:

$$\text{grad } p = -(\xi p)\xi, \ \text{grad}\sigma = -(\xi\sigma)\xi \quad (42)$$

The relations (40) and (41) implies that p and σ are constants over a hypersurface. This leads to the following:

Theorem 3: If the energy-momentum tensor of a perfect fluid $(CS)_4$-spacetime is a Codazzi tensor, then both the energy density and pressure of the fluid are constants over a hypersurface.

Again from (2)-(6), it follows that in a $(CS)_4$-spacetime, the following relation holds:

$$(\nabla_X\eta)(Y) = \frac{1}{3}\text{div}\xi\left[g(X,Y) + \eta(X)\eta(Y)\right] \quad (43)$$

Since the integral curves of ξ in a $(CS)_4$-spacetime are geodesics[7], the Roy-Choudhuri equation[5] for the fluid in a $(CS)_4$-spacetime can be written as:

$$(\nabla_X\eta)(Y) = \omega(X,Y) + \tau(X,Y) + \frac{1}{3}\text{div}\xi\left[g(X,Y) + \eta(X)\eta(Y)\right] \quad (44)$$

where ξ is the velocity vector field of the fluid, ω is the vorticity tensor and τ is the shear tensor respectively. Comparing (43) and (44) we get:

$$\omega(X,Y) + \tau(X,Y) = 0 \quad (45)$$

Again in a $(CS)_4$-spacetime we have[7] curl $\xi = 0$ i.e., ξ is irrotational. Hence the vorticity of the fluid vanishes. Therefore $\omega(X,Y)=0$. Consequently (45) implies that $\tau(X,Y)=0$. Thus we can state the following:

Theorem 4: In a perfect fluid $(CS)_4$-spacetime, the fluid has vanishing vorticity and vanishing shear.

According to Petrov[4] classification, a spacetime can be divided into six types denoted by I, II, III, D, N and O. Again, Barnes[1] has been proved that if a perfect fluid spacetime is shear free and vorticity free and the velocity vector field is hypersurface orthogonal and the energy density is constant over a hypersurface orthogonal to the velocity vector field, then the possible local cosmological structures of the spacetime are of Petrov type I, D or O. Since in a perfect fluid $(CS)_4$-spacetime the velocity vector field ξ of the fluid is always hypersurface orthogonal[7], by virtue of Theorem 3 and Theorem 4, we can state the following:

Theorem 5; If the energy-momentum tensor of a perfect fluid $(CS)_4$-spacetime is a Codazzi tensor, then the possible local cosmological structure of the spacetime is of Petrov type I, D or O.

Again, it can be easily seen that in a $(LCS)_n$-manifold(n>3) the divergence of the conformal curvature tensor C is given by:

$$(\text{div}C)(X,Z)Z = \frac{n-3}{n-2}\begin{bmatrix}(\nabla_X S)(Y,Z) \\ -(\nabla_Y S)(X,Z)\end{bmatrix} + \frac{n}{n-2}\begin{bmatrix}dr(X)g(Y,Z) \\ -dr(Y)g(X,Z)\end{bmatrix} \quad (46)$$

Hence if a perfect fluid $(CS)_4$-spacetime is divergence free conformal curvature tensor, then (46) yields:

$$\frac{1}{2}\begin{bmatrix}(\nabla_X S)(Y,Z) \\ -(\nabla_Y S)(X,Z)\end{bmatrix} + 2\begin{bmatrix}dr(X)g(Y,Z) \\ -dr(Y)g(X,Z)\end{bmatrix} = 0 \quad (47)$$

Taking an orthonormal frame field and contracting (47) over Y and Z we obtain:

$$dr(X) = 0, \quad \text{for all } X \quad (48)$$

Using (48) in (47) we have:

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z) \quad (49)$$

This implies that the Ricci tensor is a Codazzi tensor. Using (48) and (49) in (26) we obtain (24) and hence the energy-momentum tensor is a Codazzi tensor. This leads to the following:

Theorem 6: If a perfect fluid $(CS)_4$-spacetime is of divergence free conformal curvature tensor, then its energy-momentum tensor is of Codazzi type.
Consequently by virtue of Theorem5 and Theorem 6, we can state the following:

Theorem 7: If a perfect fluid $(CS)_4$-spacetime is of divergence free conformal curvature tensor, then the possible local cosmological structure of such a spacetime is of Petrov type I, D or O.

Again, Sharma[8] proved that if a spacetime with divergence free conformal curvature tensor admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N.
This leads to the following:

Theorem 8: If a perfect fluid $(CS)_4$-spacetime with divergence free conformal curvature tensor admits a conformal Killing vector field, then the spacetime is either conformally flat or of Petrov type N.

Conformally flat $(CS)_4$-spacetimes: This section deals with a conformally flat $(CS)_4$-spacetime. It can be easily seen that in a conformally flat $(CS)_4$-spacetime, the Ricci tensor and curvature tensor are given by:

$$S(X,Y) = \left[\frac{r}{3} - (\rho - \alpha^2)\right] g(X,Y) + \left[\frac{r}{3} - 4(\rho - \alpha^2)\right] \eta(X)\eta(Y) \quad (50)$$

$$R(X,Y)Z = \left[\frac{r}{6} - (\rho - \alpha^2)\right] \begin{Bmatrix} g(Y,Z)X \\ -g(X,Z)Y \end{Bmatrix} + \frac{1}{2}\left[\frac{r}{3} - 4(\rho - \alpha^2)\right] \begin{bmatrix} \begin{Bmatrix} g(Y,Z)\eta(X) - \\ g(X,Z)\eta(Y) \end{Bmatrix} \xi \\ + \begin{Bmatrix} \eta(Y)\eta(Z)X \\ -\eta(X)\eta(Z)Y \end{Bmatrix} \end{bmatrix} \quad (51)$$

for all X, Y, Z.

Let $\xi^{\perp}$ be denote the 3-dimensional distribution in a $(CS)_4$-spacetime orthogonal to ξ. Then we have $\eta(X) = \eta(Y) = \eta(Z) = 0$ for all $X, Y, Z \in \xi^{\perp}$. Thus from (51) we have:

$$R(X,Y)Z = \left(\frac{r}{6} - \frac{\rho - \alpha^2}{2}\right) \begin{bmatrix} g(Y,Z)X \\ -g(X,Z)Y \end{bmatrix} \quad (52)$$

for all $X, Y, Z \in \xi^{\perp}$

This implies that:

$$R(X,\xi)\xi = -\left(\frac{r}{6} - \frac{\rho - \alpha^2}{2}\right) X \quad \text{for all } X \in \xi^{\perp} \quad (53)$$

Again, according to Karcher[2], a Lorentzian manifold is called infinitesimally spatially isotropic relative to a unit timelike vector field U if its Riemann curvature tensor R satisfies the relation:

$$R(X,Y)Z = \delta[g(Y,Z)X - g(X,Z)Y]$$

for all $X, Y, Z \in U^{\perp}$
and $R(X,U)U = \gamma X$ for $X \in U^{\perp}$,

where δ, γ are real valued functions on the manifold. Hence by virtue of (52) and (53), we can state the following:

Theorem 9: A conformally flat $(CS)_4$-spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field ξ.

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